Higher Complex Structures

Master's Thesis

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Abstract

In this report, we introduce and analyze a new structure on surfaces generalizing the complex structure. To define this so called higher complex structure we need the punctual Hilbert scheme of the plane which roughly speaking gives a polynomial curve in each cotangent space. In the case where these curves are straight lines, we recover the complex structure. We show that the higher complex structure is locally trivializable by higher diffeomorphisms, a generalization of usual diffeomorphisms. The global theory gives an interesting generalization of the classical Teichmüller space. We hope that this approach will give a geometric version of higher Teichmüller theory.

Contents

1 Introduction

Manifolds are the main objects in differential geometry. Because of their richness, it is often convenient to equip them with some extra structure. The local and global theory of extra structures often reveal topological properties of the manifold.

Motivation. We are interested in structures related to the complex structure. A complex structure is a complex atlas. Its global theory gives the notion of the Teichmüller space $\mathcal{T}_\Sigma$ which describes all complex structures modulo diffeomorphisms isotopic to the identity.

Poincaré's uniformization theorem links complex structures on surfaces to homomorphisms from the fundamental group of the surface to $PSL(2,\mathbb{R})$, the automorphism group of the hyperbolic plane. With this, the Teichmüller space can be seen as a connected component of the space of representations of the fundamental group in $PSL(2,\mathbb{R})$ with discrete image:

$$\mathcal{T}_\Sigma \subset \mathrm{Rep}_{\mathrm{discrete}}(\pi_1(\Sigma), PSL(2,\mathbb{R})).$$

Hitchin describes in [16] an algebraic generalization of the Teichmüller space in terms of representations of $\pi_1(\Sigma)$ in $PSL(n,\mathbb{R})$. The geometric nature of this generalization remains mysterious: Does this generalization describe the global theory of a structure on the surface similar to the complex structure? In this report, we describe a candidate for a new structure on surfaces generalizing the complex structure. We hope that this so called higher complex structure will give a geometric approach to higher Teichmüller theory.

Outline. The report is structured in three parts.

In the first part in section 2, we will see several viewpoints of the complex structure on surfaces. Starting with the definition, we analyze the almost complex structure, especially its local and global theory. Then, we discuss Poincaré's uniformization theorem and its consequences for the global theory of complex structures. In a last part, we link to complex projective structures. All this material is classic. We will see that in preparation for our generalization, you should think of the complex structure as a given direction in each complexified cotangent space.

In the second part (section 3), we introduce the tool we need for generalizing the complex structure: the punctual Hilbert scheme of the plane. This scheme can be variously seen as the set of ideals of $\mathbb{C}[x,y]$ of finite codimension, a blowup of the configuration space $\mathrm{Sym}^n(\mathbb{C}^2)$ or the set of commuting matrices admitting a cyclic vector. Roughly speaking, the punctual Hilbert scheme describes sets of points of the plane and whenever two or more points coincide, it keeps track of how they collide.

We then analyze the zero-fiber of the Hilbert scheme which corresponds to the case when all points coincides. In this case, we can simply think of it as a polynomial curve. This part is also well-known apart from the two last subsections in which we rediscover Lagrange's inversion theorem and explore an orthogonal viewpoint.

In the third part (section 4), we define and analyze the higher complex structure. To gain more flexibility, we will enlarge the group of diffeomorphisms and look at higher complex structures modulo higher diffeomorphisms isotopic to the identity. We will show that the local theory is trivial as for the complex structure. The global theory leads to a geometric generalization of the Teichmüller space. We will describe its tangent and cotangent space and compute its dimension. All the material in this section is new, although the approach was strongly suggested to me by my advisor.

Subsections marked with a star are not important for the global understanding and are not refered to in subsequent sections.

Prerequisites. The reader is supposed to be familiar with basic facts about differential and complex geometry (that's important), algebraic geometry (if you want to understand details in section 3) and symplectic geometry (for section 4, we will recall the definitions). Manuels for these topics are the books of Lee [21], Griffiths & Harris [13] and McDuff & Salamon [22].

Notations. Throughout the paper, M denotes a smooth manifold and Σ a connected oriented surface (real dimension 2) of genus g, often required to be at least 2. A complex coordinate on Σ is denoted by $z = x+iy$ and its conjugate coordinate on $T^*\Sigma$ by p.
Other notations: $\mathbb{H}$ is the hyperbolic plane, J an almost complex structure, ∂ and $\bar{\partial}$ defined on page 5, μ and μ_n the Beltrami coefficient and higher Beltrami coefficients, $\propto$ means "proportional to", $K = T^{*(1,0)}\Sigma$ is the canonical line bundle, $\Gamma(L)$ the global sections of some line bundle L, $\mathcal{T}_\Sigma$ and $\hat{\mathcal{T}}^n_\Sigma$ the Teichmüller space and its geometric generalization, $\mathcal{P}_\Sigma$ defined on page 12, $S(w,z)$ the Schwarzian derivative, Hilb^n and Hilb^n_0 the punctual Hilbert scheme of the plane and its zero-fiber, $\mathrm{Sym}^n(\mathbb{C}^2)$ the configuration space (page 18), $\nu \vdash n$ a partition of n (page 25 and 26), c_ν and ε_ν defined on page 26 and 27, $\{.,.\}$ the Poisson bracket (p. 37) and Tf defined on page 41.

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I'm also thankful to John Baez and Javier Muniain for their marvelous book *Gauge Fields, Knots and Gravity* [3], the best book I ever read and from which I take endless motivation for my research.

2 Complex structures on surfaces

In this section we will discuss various viewpoints of a complex structure on a surface, its main properties and the associated Teichmüller space. Of course, our presentation will prepare the reader to the generalization which will follow in section 4.

2.1 Complex structure

Once mathematicians understood the universal and deep properties of complex numbers and their surprising appearance in physical theories, it was natural to generalize most of the mathematical concepts to complex numbers. Thus, it is not surprising that the fundamental notion in geometry, that of a manifold, was generalized to a complex manifold.

Definition 1. *A **complex structure** on a manifold M of real dimension $2n$ is an atlas with coordinate charts being open subsets of $\mathbb{C}^n$ such that the transition functions are holomorphic. A manifold with a complex structure is called a **complex manifold**.*

Just as real differential manifolds are modeled on $\mathbb{R}^n$, complex manifolds are modeled on $\mathbb{C}^n$. Simple examples of complex manifolds are given by open sets of $\mathbb{C}^n$ or the complex projective spaces $\mathbb{C}P^n$.

For terminology, a complex manifold of dimension one (i.e. a complex curve) is called a **Riemann surface**. Riemann surfaces have been studied intensely which permitted among others the understanding of multivalued functions like for instance the complex square root as ramified covering map over $\mathbb{C}$. My favorite reference for Riemann surfaces is the book of Donaldson [7]. A nice book on complex manifolds is Kodaira's book [20].

2.2 Almost complex structure

Adding a supplementary structure on a manifold can basically be done in two ways: *either one adds a geometric object to the manifold or one changes the model space.* Often, there are two notions corresponding to the two approaches and a *deep understanding of the structure involves a link between these two approaches.* We will encounter several examples of this principle: almost complex, hyperbolic and symplectic structure. For all of them, one adds a geometric object to the manifold but deep theorems (Newlander-Nirenberg, Killing-Hopf and Darboux resp.) give that they can be modeled on $\mathbb{C}^n$, $\mathbb{H}^n$ and $(\mathbb{R}^{2n}, \omega_0)$ respectively.

In our case, we have defined a complex manifold by changing the model space. Keeping only the fact that every tangent space carries a structure of a complex vector space, one gets the notion of an almost complex structure:

Definition 2. *An* ***almost complex structure*** *on a smooth manifold M is an endomorphism $J(m)$ on T_mM for all points m of M depending smoothly on the point and satisfying $J(m)^2 = -\operatorname{id}$.*

This definition clearly extends the property $i^2 = -1$. More precisely, a complex structure on a manifold gives an obvious almost complex structure by pulling back the multiplication by i on M. An almost complex structure is said to be **integrable** if it comes from a complex structure in this way. Note that a manifold with an almost complex structure has necessarily even dimension by taking the determinant of the equation $J^2 = -\operatorname{id}$.

The study of an additional structure on a manifold is always done in three steps: first one has to understand the structure *for one point* which often is linear algebra, then *locally in a neighborhood of a point* and finally *globally*. From now on, we restrict attention to surfaces.

Let's start with the first step: An almost complex structure J on $\mathbb{C}$ is uniquely determined by the image of 1 which has to be in $\mathbb{C}-\mathbb{R}$. Thus, we see that the space of almost complex structures on $\mathbb{C}$ has two connected components both homeomorphic to the upper half plane $\mathbb{H}$. These two components are canonically isomorphic since we can associate an almost complex structure $-J$ to any J. Any almost complex structure gives an orientation (on every tangent space so on the manifold) by declaring $(1, J(1))$ to be a direct basis. That's why we always restrict attention to oriented surfaces with almost complex structures **compatible with the orientation**, meaning that the orientation coming from J coincides with the surface orientation.

For the second step, a miracle happens already discovered, as lots of other miracles, by Carl-Friedrich Gauss.

Theorem 1 (Gauss 1822 (real analytic case), Korn and Lichtenstein 1916)**.** *Any almost complex structure on a surface is integrable, i.e. locally trivialisable.*

This theorem is equivalent to the existence of **isothermal coordinates**, i.e. coordinates such that the metric locally reads $g = f(x,y)(x^2+y^2)$. A proof of this can be found in [6].

The theorem does not hold true in higher dimensions. In fact, it is possible to associate a notion of curvature to an almost complex structure. The famous **theorem of Newlander and Nirenberg** asserts that *an almost complex structure* *is integrable if and only if its curvature is vanishing.* In the case of a real analytic manifold, this boils down to the Frobenius theorem on integrable distributions.

The third step, the global understanding of a complex structure, leads to the notion of the Teichmüller space. Before going there, we need a careful analysis of the endomorphism J which will be done in terms of the Beltrami coefficient.

2.3 Beltrami coefficient

To study an almost complex structure, we wish to diagonalize the endomorphism J. Since $J^2 = -\mathrm{id}$, the characteristic polynomial is given by $X^2 + 1$, so the eigenvalues of J are i and $-i$. Thus, we have to complexify the vector space $T\Sigma$ into $T^{\mathbb{C}}\Sigma := TM \otimes_{\mathbb{R}} \mathbb{C}$. Note that the complexified tangent space carries a "natural" complex multiplication by i coming from the complexification and has also the almost complex structure J (extended by $\mathbb{C}$-linearity). We then get the decomposition

$$T^{\mathbb{C}}\Sigma = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$$

with $T^{1,0}\Sigma$ the eigenspace of J associated to the eigenvalue i and $T^{0,1}\Sigma$ to $-i$.

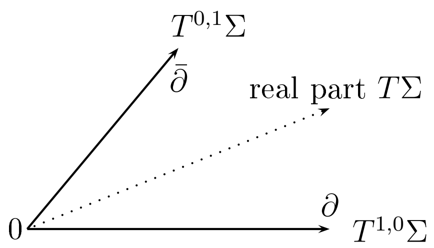

Fig. 1: Complexified tangent space

Explicitly, we have

$$T^{1,0}\Sigma = \{X - iJX \mid X \in T\Sigma\} \text{ and } T^{0,1}\Sigma = \{X + iJX \mid X \in T\Sigma\}$$

since for example $J(X - iJX) = JX - iJJX = i(X - iJX)$.

In the case of an almost complex structure J_0 coming from a complex chart $z = x + iy$, put

$$\partial = \partial_z := \frac{1}{2}(\partial_x - i\partial_y) \text{ and } \bar{\partial} = \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y).$$

We see that $T^{1,0}\Sigma$ (resp. $T^{0,1}\Sigma$) is generated by ∂ (resp. $\bar{\partial}$). The differential operator $\bar{\partial}$ is called **Cauchy-Riemann operator**.

For any other almost complex structure J, we can express its eigenvectors in the basis formed by $(\partial, \bar{\partial})$ of $T^{\mathbb{C}}\Sigma$ coming from some fixed complex chart. Say for instance that an eigenvector for $-i$ is given by $\chi := a\partial + b\bar{\partial}$ with $a, b \in \mathbb{C}$. Then, an eigenvector for i is given by $\bar{\chi} = \bar{a}\bar{\partial} + \bar{b}\partial$ where the complex conjugation comes from the natural complex structure on $T^{\mathbb{C}}\Sigma$.

Since we are on a surface, the endomorphism J is uniquely determined by its two eigenspaces generated by the vectors χ and $\bar{\chi}$. We only have to ensure that these two vectors are linearly independent since their corresponding eigenvalues are not the same. A direct computation shows that $\chi = a\partial + b\bar{\partial}$ is linearly independent to $\bar{\chi} = \bar{a}\bar{\partial} + \bar{b}\partial$ if and only if $a\bar{a} \neq b\bar{b}$. In the case where $b \neq 0$, we can normalize χ to $\chi = \bar{\partial} - \mu\partial$ with $\mu = -a/b \in \mathbb{C}$. This number μ is called the **Beltrami coefficient**. The condition $a\bar{a} \neq b\bar{b}$ reads $\mu\bar{\mu} \neq 1$. If $J = J_0$ then $\mu = 0$.

In the first step, we have seen that almost complex structures come in pairs (J and $-J$). The Beltrami coefficient of $-J$ is $1/\bar{\mu}$. We can thus restrict attention to the case $|\mu| < 1$.

An almost complex structure is encoded by the Beltrami coefficient. For the moment, we just saw an expression in a local chart. To understand the global nature of μ we have to look how it transforms under coordinate changes. Changing the holomorphic chart z to w given by $z \mapsto z(w)$, the partial derivatives become $\frac{\partial}{\partial z} \mapsto \frac{dw}{dz}\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial \bar{z}} \mapsto \frac{d\bar{w}}{d\bar{z}}\frac{\partial}{\partial \bar{w}}$. Thus,

$$\partial_{\bar{z}} - \mu(z,\bar{z})\partial_z \mapsto \frac{d\bar{w}}{d\bar{z}}\partial_{\bar{w}} - \mu(z,\bar{z})\frac{dw}{dz}\partial_w \propto \partial_{\bar{w}} - \frac{d\bar{z}/d\bar{w}}{dz/dw}\mu(z,\bar{z})\partial_w$$

where $\propto$ simply means that the two vectors are proportional.
Thus,

$$\mu(z,\bar{z}) \mapsto \mu(w,\bar{w}) = \frac{d\bar{z}/d\bar{w}}{dz/dw}\mu(z,\bar{z}). \tag{1}$$

We say that μ is of type (-1,1) which means that it is a section of $K^{-1} \otimes \bar{K}$ where $K = T^{*(1,0)}\Sigma$ denotes the canonical bundle of Σ and $K^{-1} = K^*$ in the **Picard group**, the group of line bundles over a Riemann surface with tensor product as composition. More explicitly, the Beltrami coefficient is an object of the form $\mu(z,\bar{z})d\bar{z} \otimes \partial$, i.e. a (0,1)-form with values in K:

$$\mu \in \Omega^{0,1}(\Sigma, K) = \Gamma(K^* \otimes \bar{K}).$$

Remark. *The variable $\bar{z}$ has no geometric meaning. Writing $\mu(z,\bar{z})$ simply indicates that μ is not necessarily holomorphic.*

A fundamental principal in algebraic geometry is to *understand a space via its functions defined on it.* In this spirit, we define a **holomorphic function** f (with respect to the almost complex structure J) by a function satisfying the **Beltrami equation**

$$(\bar{\partial} - \mu\partial)f = 0$$

which is a generalization of the usual Cauchy-Riemann equation. In the case of the natural complex structure J_0, the Beltrami coefficient vanishes and one recovers the Cauchy-Riemann equation.
Conversely, given any holomorphic function f with respect to J, we can recover the Beltrami coefficient by the formula

$$\mu = \frac{\bar{\partial} f}{\partial f}$$

which gives another explanation for the expression $\mu(z, \bar{z})d\bar{z} \otimes \partial$.

Remark. *Change under general transformation*
Under a general transformation $z \mapsto z(w, \bar{w})$ (not necessary holomorphic), we get $\frac{\partial}{\partial z} \mapsto \frac{\partial w}{\partial z}\frac{\partial}{\partial w} + \frac{\partial \bar{w}}{\partial z}\frac{\partial}{\partial \bar{w}}$ and analogously for $\frac{\partial}{\partial \bar{z}}$. Thus,

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} - \mu(z, \bar{z})\frac{\partial}{\partial z} &\mapsto \left(\frac{\partial w}{\partial \bar{z}} - \mu(z, \bar{z})\frac{\partial w}{\partial z}\right)\frac{\partial}{\partial w} + \left(\frac{\partial \bar{w}}{\partial \bar{z}} - \mu(z, \bar{z})\frac{\partial \bar{w}}{\partial z}\right)\frac{\partial}{\partial \bar{w}} \\ &\propto \frac{\partial}{\partial \bar{w}} - \frac{-\frac{\partial w}{\partial \bar{z}} + \mu(z, \bar{z})\frac{\partial w}{\partial z}}{\frac{\partial \bar{w}}{\partial \bar{z}} - \mu(z, \bar{z})\frac{\partial \bar{w}}{\partial z}}\frac{\partial}{\partial w} \end{aligned}$$

Thus,

$$\mu(w, \bar{w}) = \frac{-\frac{\partial w}{\partial \bar{z}} + \mu(z, \bar{z})\frac{\partial w}{\partial z}}{\frac{\partial \bar{w}}{\partial \bar{z}} - \mu(z, \bar{z})\frac{\partial \bar{w}}{\partial z}}.$$

This homographic change shows that μ does not live in $\mathbb{C}$ but rather in $\mathbb{C}P^1$. For a holomorphic transform, we have $\frac{\partial \bar{w}}{\partial z} = 0 = \frac{\partial w}{\partial \bar{z}}$ so we recover formula (1) *above.*

2.4 Teichmüller space

2.4.1 Definition

Now we are ready for the third step, the global understanding of complex structures.

Definition 3. *The **Teichmüller space** of an oriented manifold M denoted by $\mathcal{T}_M$ is the set of all complex structures on M compatible with the orientation divided by all diffeomorphisms isotopic to the identity.*

The definition means that in the Teichmüller space, two complex structures are considered as the same when one is the pullback by a diffeomorphism isotopic to the identity of the other. The space of compatible complex structures modulo all diffeomorphisms is called **moduli space**. Since this space is singular (for surfaces it is an orbifold), we prefer considering the Teichmüller space.

The following famous theorem discovered by Teichmüller and proved by Ahlfors and Bers gives the global picture for complex structures on surfaces:

Theorem 2 (Teichmüller, Ahlfors, Bers)**.** *The Teichmüller space of a surface* Σ *of genus* $g \geq 2$ *is a contractible manifold of complex dimension* $3g-3$.

A quite accessible proof of this theorem using the *pants decomposition* can be found in [2]. Another proof is given in [10] using laminations. We will give a third proof in which we show even a little bit more since we will describe the tangent and cotangent space of $\mathcal{T}_\Sigma$ with our analysis of the Beltrami coefficient. In particular we show that it is a smooth manifold and recover its dimension.

Note that contractibility is easy since a compatible complex structure is given by the Beltrami coefficient μ satisfying $|\mu| < 1$. Considering $(1-t)\mu$ for $t \in [0,1]$ gives a retraction of the Teichmüller space to the trivial structure $\mu = 0$.

2.4.2 Tangent and cotangent space of $\mathcal{T}_\Sigma$

We will give an explicit description of the tangent and cotangent space to the Teichmüller space and recover in particular its dimension.

Theorem 3. *The cotangent space to the Teichmüller space at any point* J *is the set of all holomorphic quadratic differentials.*

Proof. We have seen that the connected component of the set of complex structures at one point is $\{\mu \in \mathbb{C} \mid \mu\bar{\mu} < 1\}$. Thus, we have

$$\mathcal{T}_\Sigma = \{\mu \in K^{-1} \otimes \bar{K} \mid |\mu| < 1\} / \operatorname{Diff}_0 \Sigma$$

where $\operatorname{Diff}_0 \Sigma$ denotes all diffeomorphisms isotopic to the identity. Thus,

$$T_J(\mathcal{T}_\Sigma) = \{\delta\mu \in K^{-1} \otimes \bar{K}\} / \Gamma(T\Sigma) \tag{2}$$

where $J \in \mathcal{T}_\Sigma$ and $\Gamma(T\Sigma)$ denotes the space of all smooth vector fields on Σ.

All we have to do is to calculate the action of a (real) vector field $\chi = v\partial + \bar{v}\bar{\partial}$ on the Beltrami coefficient μ. By theorem 1, there is an atlas in which $\mu = 0$. So we are interested in variations $\delta\mu$ around $\mu = 0$ coming from vector fields.

Recall that a "variation" is nothing else than the derivative at 0 in some direction: $\delta\mu = \frac{d}{dt}\mu(z+tz_0)\big|_{t=0}$.

It is well known that the variation of a vector field V by another vector field W is given by $\mathcal{L}_W V = [W,V]$ where $\mathcal{L}$ denotes the Lie derivative and $[.,.]$ the Lie bracket. Thus, the variation of $\bar\partial$ by χ is given by

$$[\chi, \bar\partial] = [v\partial + \bar v\bar\partial, \bar\partial] = -(\bar\partial v)\partial - (\bar\partial \bar v)\bar\partial.$$

Integrating the vector field χ gives a one parameter family indexed by a real number ε. Thus,

$$\bar\partial \mapsto \bar\partial - \varepsilon((\bar\partial v)\partial + (\bar\partial\bar v)\bar\partial) \propto \bar\partial - \frac{\varepsilon\bar\partial v}{1-\varepsilon\bar\partial\bar v}\partial = \bar\partial - \varepsilon(\bar\partial v)\partial$$

at first order in ε.
Since $\mu \mapsto \mu + \varepsilon\delta\mu$ by definition of $\delta\mu$, we get

$$\delta\mu = \bar\partial v \tag{3}$$

Notice that for a holomorphic vector field χ, we have $\bar\partial v = 0$ so the complex structure does not change.
Finally, we get from (2) that

$$T_J(\mathcal{T}_\Sigma) = \{\delta\mu \in K^{-1}\otimes\bar K\}/\bar\partial v$$

For the cotangent space $T_J^*(\mathcal{T}_\Sigma)$, we get even a simpler form using the fact that the dual space to $K^{-1}\otimes\bar K$ is K^2 (the pairing being integration over Σ):

$$\begin{aligned}
T_J^*(\mathcal{T}_\Sigma) &= (\{\delta\mu \in K^{-1}\otimes\bar K\}/\bar\partial v)^* \\
&= \mathrm{Ann}\,(\{\bar\partial v\}) \\
&= \{t \in K^2 \mid \textstyle\int t(\bar\partial v) = 0\ \forall v \in K^{-1}\} \\
&= \{t \in K^2 \mid \textstyle\int v(\bar\partial t) = 0\ \forall v \in K^{-1}\} \\
&= \{t \in K^2 \mid \bar\partial t = 0\} \\
&= H^0(\Sigma, K^2)
\end{aligned}$$

the last space being the 0-th (Čech) cohomology group which is nothing else than the set of holomorphic sections of the line bundle K^2 over Σ, i.e. the set of holomorphic quadratic differentials. □

Let's recover the dimension of the Teichmüller space. The **Riemann-Roch formula**

$$\dim_{\mathbb{C}}(H^0(L)) - \dim_{\mathbb{C}}(H^1(L)) = \deg L - g + 1$$

where L is a complex line bundle over a Riemann surface Σ of genus g, coupled with **Serre duality**

$$H^1(L) \cong H^0(L^{-1} \otimes K),$$

gives for $L = K^2$ that

$$\dim_{\mathbb{C}} H^0(K^2) - \dim_{\mathbb{C}} H^0(K^{-1}) = 2(2g-2) - g + 1 = 3g - 3$$

since $\deg K^2 = 2\deg K = -2\chi(\Sigma) = 2(2g-2)$ where $\chi(\Sigma)$ is the Euler characteristic. For $g \geq 2$, we have $\deg K^{-1} = 2 - 2g < 0$ so there is no global non-zero holomorphic section since the degree is the number of zeros minus the number of poles (with multiplicity) of any meromorphic section. Thus for $g \geq 2$, we get $\dim_{\mathbb{C}} H^0(K^2) = 3g - 3$.
Therefore

$$\dim_{\mathbb{C}} \mathcal{T}_\Sigma = \dim_{\mathbb{C}} T^*_J(\mathcal{T}_\Sigma) = \dim_{\mathbb{C}} H^0(K^2) = 3g - 3.$$

Remark. *From formula* (3)*, we see that the tangent space to the Teichmüller space is the cokernel of the map*

$$\bar\partial : \Omega^0(\Sigma, K) \to \Omega^{0,1}(\Sigma, K).$$

Remark. *Variation around an arbitrary* μ
Following the previous argument, we compute the variation $\delta\mu$ *induced by a vector field* $\chi = v\partial + \bar v\bar\partial$ *around an arbitrary* μ*.*
The infinitesimal variation of $\bar\partial - \mu\partial$ *by* χ *is given by*

$$[v\partial + \bar v\bar\partial, \bar\partial - \mu\partial] = (-\bar\partial v + \mu\partial v - v\partial\mu - \bar v\bar\partial\bar\mu)\partial + (-\bar\partial\bar v + \mu\partial\bar v)\bar\partial.$$

Thus, we get

$$\begin{aligned}\bar\partial - \mu\partial &\mapsto (1 + \varepsilon(-\bar\partial\bar v + \mu\partial\bar v))\bar\partial - (\mu + \varepsilon(\bar\partial v - \mu\partial v + v\partial\mu + \bar v\bar\partial\bar\mu))\partial \\ &\propto \bar\partial - (\mu + \varepsilon(\bar\partial v - \mu\partial v + v\partial\mu + \bar v\bar\partial\mu + \mu\bar\partial\bar v - \mu^2\partial\bar v))\partial\end{aligned}$$

Thus, noticing a nice factorization, we get

$$\delta\mu = (\bar\partial - \mu\partial + \partial\mu)(v + \mu\bar v) \tag{4}$$

2.5 Uniformization theorem

Another way to analyze the global theory of complex structures on surfaces is given by Poincaré's famous **uniformization theorem** which gives the surface as a quotient of the hyperbolic plane.

Theorem 4 (Poincaré, Koebe, 1907)**.** *Every simply connected Riemann surface is biholomorphic to either the Riemann sphere, the complex plane or the hyperbolic plane.*

The original proof of Poincaré can be found in [25]. A historical account which also gives an idea of the proof is given in [1].

As a corollary, we get that for a surface of genus $g \geq 2$, its universal cover is biholomorphic to the hyperbolic plane. Indeed, the fundamental group of Σ acts freely by automorphisms on its universal cover. Any orientation-preserving automorphism of the sphere has a fixed point and any subgroup of the automorphism group of the complex plane is abelian. Since $\pi_1(\Sigma)$ is non-abelian for $g \geq 2$, the universal cover has to be the hyperbolic plane. Since the automorphism group of the hyperbolic plane is $PSL(2, \mathbb{R})$, we get

Corollary. *A complex structure on a surface of genus $g \geq 2$ is uniquely determined by a homomorphism of its fundamental group to $PSL(2, \mathbb{R})$ up to conjugacy.*

In this point of view, the Teichmüller space $\mathcal{T}_\Sigma$ is included in the space of all representations of the fundamental group to $PSL(2, \mathbb{R})$ with discrete image:

$$\mathcal{T}_\Sigma \subset \mathrm{Rep}_{\text{discrete}}(\pi_1(\Sigma), PSL(2, \mathbb{R}))$$

where Rep denotes the set of homomorphisms modulo conjugacy. It is a theorem that $\mathcal{T}_\Sigma$ is in fact a connected component of that space (see [12]).

In [16], Hitchin proves that there is a connected component $\mathcal{T}^n_\Sigma$ of

$$\mathrm{Rep}_{\text{discrete}}(\pi_1(\Sigma), PSL(n, \mathbb{R}))$$

which generalizes the ordinary Teichmüller space in an algebraic way. For the moment, the only geometric meaning of this special component is given by generalizing the hyperbolic structure, for instance in [14] for $n = 4$. But there is no interpretation of $\mathcal{T}^n_\Sigma$ as the moduli space of a structure similar to the complex structure. We hope that our geometric higher Teichmüller space (see section 4.4), coming from the global theory of higher complex structures, will give such an interpretation.

Another spin-off coming from the uniformization theorem is that every Riemann surface of genus $g \geq 2$ admits a **hyperbolic structure**. A hyperbolic structure on a manifold is a Riemannian metric with constant negative sectional curvature. The Killing-Hopf theorem gives that this is equivalent to changing the model space $\mathbb{R}^n$ to $\mathbb{H}^n$ with transition functions in $SO(1,n)$, the group of isometries of the hyperbolic space. Therefore, every surface with hyperbolic structure has naturally a complex structure since $\mathbb{H} \subset \mathbb{C}$ and $SO(1,2) \cong PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ and homographies are holomorphic.

2.6 Complex projective structure*

In this subsection, we discuss another structure which is related to the global theory of the complex structure.

Always guided by our principle that a new structure is either an additional geometric object or a change of the model space, we define a **complex projective structure** to be an atlas with charts in $\mathbb{C}P^n$ and transition functions in $PSL(n+1,\mathbb{C})$, the group of automorphisms of the complex projective space. Two complex projective manifolds M_1 and M_2 are isomorphic if there is a diffeomorphism $f : M_1 \to M_2$ that pulls back projective charts of M_2 to projective charts of M_1. The analogue of the Teichmüller space in this setting, the space of all complex projective structures modulo all automorphisms isotopic to the identity, is denoted by $\mathcal{P}_M$. A survey on this structure can be found in [8].

We are interested in the case $n = 1$. Since $\mathbb{C}P^1$ is a complex manifold and since any homography is holomorphic, a projective structure gives a complex structure. So a projective structure is a finer notion but the equivalence relation is stronger too. So, once we pass to equivalence classes, the space $\mathcal{P}_\Sigma$ is actually bigger than the Teichmüller space:

Theorem 5. *There is a surjective map $\pi : \mathcal{P}_\Sigma \to \mathcal{T}_\Sigma$ with fiber given by the set of all holomorphic quadratic differentials $H^0(\Sigma, K^2)$.*

We will give a proof following indications of my advisor. This proof is not new, see for instance [8] (section 3.2).

Let's give first an idea of the proof. The surjectivity of the map is essentially given by the uniformization theorem. To compute the inverse image of a point, we show that a projective structure on a surface is equivalent to a differential operator of order 2 on the surface modulo some action of functions. Such an operator can be reduced to the form $\partial^2 + t$ where the difference of two different t's is a holomorphic quadratic differential. Since a complex structure is uniquely determined by the operator ∂, the fiber is given by $H^0(\Sigma, K^2)$.

Proof. Since a projective structure gives a complex structure, the map is well-defined by taking the quotient by all diffeomorphisms isotopic to the identity.

By the uniformization theorem, a surface Σ with given complex structure is biholomorphic to a quotient of the hyperbolic plane. For such a quotient, the transition functions are in $SO(1,2) \cong PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$. This gives surjectivity.

As announced, we will link projective structures to differential operators of order 2. To any operator $D = a\partial^2 + b\partial + c$ we associate a projective structure in the following way: We know that locally, the space of solutions to $D\psi = 0$ is a two-dimensional vector space. Choose a basis (ψ_1, ψ_2) and put $u := \psi_1/\psi_2$. The function u changes by a homography when we change the basis (ψ_1, ψ_2). Thus, u can be seen as a chart to $\mathbb{C}P^1$ which gives the projective structure.

Notice that the space of all smooth non-vanishing functions $\mathcal{O}^*$ acts on differential operators on the left and also on the right. In this case, both actions don't change the map u. Indeed, for a smooth non-vanishing function f, $D\psi = 0$ implies $(fD)\psi = f(D\psi) = 0$ so nothing changes by left action. By right action, define $\psi_i' = \frac{1}{f}\psi_i$ for $i = 1, 2$, then $D\psi_i = 0$ implies $(Df)\psi_i' = D(f\psi') = D\psi = 0$, so $u' = \psi_1'/\psi_2' = \psi_1/\psi_2 = u$.

We will prove that the map from the set of all differential operators of order 2 modulo the right and left action of $\mathcal{O}^*$ to $\mathcal{P}_\Sigma$ is an isomorphism. For that, we will show that given any local chart ϕ to an open subset of $\mathbb{C}P^1$, there is an unique (up to $\mathcal{O}^*$-action) operator D of order 2 satisfying $D\phi = 0$ and $D(1) = 0$. In that case, we have $u = \phi$. To do so, we will first give a reduced form for any D using the left and right action of $\mathcal{O}^*$. This reduced form will be $\partial^2 + t$.

We start with $D = a\partial^2 + b\partial + c$. Absorbing the coefficient a by a function on the left, we can assume $a = 1$. We are looking for a function g such that Dg is proportional to $\partial^2 + t$. A direct computation gives

$$Dg(\psi) = D(g\psi) = g\partial^2\psi + (gb + 2\partial g)\partial\psi + (gc + \partial^2 g + b\partial g)\psi$$

So we choose g such that $gb + 2\partial g = 0$ which is always possible. We then have $\partial g = -b/2g$ and $\partial^2 g = g/2(b^2/2 - \partial b)$ and thus,

$$\frac{1}{g}Dg = \partial^2 + (c - \frac{b^2}{4} - \frac{\partial b}{2}).$$

This shows the existence of the reduced form. Uniqueness can be seen directly.

Now, we are looking for a solution of $D(1) = 0$ and $D\phi = 0$ where $D = \partial^2 + b\partial + c$. The first equation simply gives $c = 0$. The second equation gives $b = -\frac{\partial^2\phi}{\partial\phi}$. So

$$D = \partial^2 - \frac{\partial^2\phi}{\partial\phi}\partial$$

and the reduction process gives $D = \partial^2 + t$ with

$$t = \frac{1}{2}\left(\frac{\partial^3\phi}{\partial\phi} - \frac{3}{2}(\frac{\partial^2\phi}{\partial\phi})^2\right).$$

Twice the term on the right sight is known as the **Schwarzian derivative**, denoted by $S(\phi, z)$.

Therefore, we see that modulo the action of $\mathcal{O}^*$, there is a unique operator D giving $u = \phi$ which gives the isomorphism[1]

$$\mathcal{P}_\Sigma \cong \{D\}/\mathcal{O}^* = \{\partial^2 + t\}.$$

All what's left is to understand the global nature of t. So we have to see how t changes under coordinate transforms.

For a coordinate change $z \mapsto z(w)$, we have $\frac{\partial}{\partial z} \mapsto \frac{dw}{dz}\frac{\partial}{\partial w}$ and $\frac{\partial^2}{\partial z^2} \mapsto (\frac{dw}{dz})^2\frac{\partial^2}{\partial w^2} + \frac{d^2w}{dz^2}\frac{\partial}{\partial w}$. Thus,

$$D = \frac{\partial^2}{\partial z^2} + t(z) \mapsto (\frac{dw}{dz})^2\frac{\partial^2}{\partial w^2} + \frac{d^2w}{dz^2}\frac{\partial}{\partial w} + t(z)$$

The reduction process gives after some direct computations

$$t(w) = (\frac{dz}{dw})^2(t(z) - \frac{1}{2}S(w, z))$$

with $S(w, z)$ the Schwarzian derivative of w with respect to z.

We see that t does not transform as a tensor but just as for the Christoffel symbols in differential geometry, the difference of two t's transforms like a holomorphic quadratic differential. So given a complex structure in form of ∂, choose a projective structure which is mapped to the complex structure by π (we already know that π is surjective). The projective structure corresponds to an operator $D_0 = \partial^2 + t_0$. Any other point above the complex structure is of the form $\partial^2 + t = D_0 + (t - t_0)$ so the fiber of π is given by the holomorphic quadratic differentials. □

Remark. *The Schwarzian derivative plays an important role in complex and projective geometry since it is invariant under homographies.*

[1]In fact, we proved only the local version, for the global statement it is necessary to consider differential operators on non-trivial line bundles.

2.7 Summary

In this first part, we saw that the complex, almost complex and hyperbolic structures are equivalent on a surface. Indeed, the theorem of Gauss gives the equivalence between complex and almost complex structure and Poincaré's uniformization theorem gives the connection to the two other structures. These various viewpoints on the complex structure explain why multiple generalizations are possible.

The hyperbolic and complex projective structures are "rigid" in the sense that the group of local structure preserving maps ($SO(1,n)$ and $PSL(n+1,\mathbb{C})$ resp.) is finite-dimensional. In contrast, the group of local diffeomorphisms preserving a complex structure is the set of all holomorphic functions with non-vanishing derivative which is infinite-dimensional. We aim to generalize the complex structure by keeping an infinite-dimensional group of local structure preserving maps.

We saw that a complex structure on a surface is uniquely determined by giving a direction ∂ in the complexified tangent space $T_z^{\mathbb{C}}\Sigma$ for every point z. For our generalization, we need to replace the tangent by the cotangent space (to gain a canonical symplectic structure). Since J acts also on the complexified cotangent space, a complex structure is also given by a direction in $T_z^{*\mathbb{C}}\Sigma$. Thus, a complex structure can be seen as a section of the (pointwise) projectivized cotangent space $\mathbb{P}(T^{*\mathbb{C}}\Sigma)$. Incidentally, we have for any vector space V that $\mathbb{P}(V) = \mathrm{Hilb}_0^2(V)$ where Hilb_0^2 is the zero-fiber of the punctual Hilbert scheme which leads miraculously to the next section.

3 The punctual Hilbert scheme

The **Hilbert scheme** is the parameter space of all subschemes of an algebraic variety. In general it is a quite complicated scheme but we are only interested in the punctual Hilbert scheme of the plane which turns out to be irreducible and smooth. An excellent account on the punctual Hilbert scheme can be found in Haiman's paper [15]. A more technical reference is [24]. We give here several viewpoints of the punctual Hilbert scheme. For understanding the higher complex structure, it is sufficient to read the paragraphs 3.1 to 3.4.

3.1 Definition and examples

Take n distinct points in the plane. We can consider these points as an algebraic variety. Its function space is of dimension n (one value for each point). This gives a simple example of a scheme of dimension zero. Such a scheme is supported on points, thus, its function space is finite-dimensional. We define the **length** of a zerodimensional scheme to be the dimension of its function space. So the variety of n distinct points is of length n. We will see that we get more interesting examples when two or several points collapses into one single point. The moduli space of zerodimensional subschemes of length n is called the punctual Hilbert scheme:

Definition 4. *The* ***punctual Hilbert scheme*** $\mathrm{Hilb}^n(\mathbb{C}^2)$ *of length n of the plane is the set of ideals of $\mathbb{C}[x,y]$ of codimension n:*

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \text{ ideal of } \mathbb{C}[x,y] \mid \dim(\mathbb{C}[x,y]/I) = n\}.$$

The subspace of $\mathrm{Hilb}^n(\mathbb{C}^2)$ consisting of all ideals supported on 0, i.e. whose associated algebraic variety is $(0,0)$, is called the ***zero-fiber*** *of the punctual Hilbert scheme and is denoted by $\mathrm{Hilb}^n_0(\mathbb{C}^2)$.*

Let's work out some examples for small values of n.

For $n=1$, the subscheme is necessarily a single point, so $\mathrm{Hilb}^1(\mathbb{C}^2) \cong \mathbb{C}^2$.

For $n=2$ we already saw the example with two distinct points. Let's see what happens when both collapses to one point (following [5]). Suppose that the first point is at $(0,0)$ and that the second point approaches along a curve $(t,\gamma(t))$ with $\gamma(0)=0$, γ holomorphic and $t \in \mathbb{C}$. The ideal defining both points at time t is given by

$$I_t = \langle x(x-t), x(y-\gamma(t)), y(x-t), y(y-\gamma(t))\rangle .$$

As t goes to 0, we see that x^2, xy and y^2 are in I_0 but these generate an ideal of codimension 3. A more thorough analysis will give a fourth element of I_0: For all

t, we see that $x(y-\gamma(t))-(x-t)y=ty-x\gamma(t)\in I_t$. Writing out the Taylor expansion of γ yields $\gamma(t)=\gamma'(0)t+\mathcal{O}(t^2)$ since $\gamma(0)=0$ and γ holomorphic. Thus, $y-x\gamma'(0)+\mathcal{O}(t)\in I_t$ for all t. When t goes to 0, we see that $y-\gamma'(0)x\in I_0$. This gives already an ideal of codimension 2. Hence

$$I_0=\left\langle x^2,-y+\gamma'(0)x\right\rangle.$$

Notice that only the slope at 0 of γ plays a role so we can choose γ to be linear. Therefore we see that the zero-fiber of the Hilbert scheme of length 2 is the projectivized plane:

$$\mathrm{Hilb}_0^2(\mathbb{C}^2)\cong\mathbb{P}(\mathbb{C}^2)=\mathbb{C}P^1$$

For $n=3$, let's collapse the points $(t,\gamma(t))$ and $(2t,\gamma(2t))$ to $(0,0)$. We will see that

$$I_0=\left\langle x^3,-y+\gamma'(0)x+\frac{\gamma''(0)}{2}x^2\right\rangle.$$

So we could choose γ to be a quadratic curve.

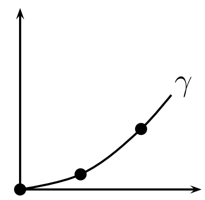

Fig. 2: Collapsing points

Some details for the interested reader: the ideal I_t at time t is given by

$$\langle x(x-t)(x-2t),x(x-t)(y-\gamma(2t)),x(y-\gamma(t))(x-2t),x(y-\gamma(t))(y-\gamma(2t)),$$
$$y(x-t)(x-2t),y(x-t)(y-\gamma(2t)),y(y-\gamma(t))(x-2t),y(y-\gamma(t))(y-\gamma(2t))\rangle$$

So when t goes to zero, we get $\langle x^3,x^2y,xy^2,y^3\rangle\subset I_0$ which only gives an ideal of codimension 6. A refinement shows that the element of I_t

$$x(x-t)(y-\gamma(2t))-y(x-t)(y-\gamma(2t))+\frac{1}{2}(x(y-\gamma(t))(x-2t)-y(y-\gamma(t))(x-2t))$$

equals

$$t^2(-y+\gamma'(0)x+\frac{\gamma''(0)}{2}x^2+\mathcal{O}(t))$$

so when t goes to zero, we get that $-y+\gamma'(0)x+\frac{\gamma''(0)}{2}x^2\in I_0$.

Remark. *If we collapse two points to the origin along two distinct straight lines, we always get $I_0=\langle x^2,xy,y^2\rangle$ which is just one point in the Hilbert scheme. We see* *that to get the generic ideal (which will be specified below), we have to approach the origin along the same curve of order 2.*

Furthermore, it is not possible to collapse first the two points outside the origin and to collapse them to the origin afterwards. This can give ideals which are too big. Thus, during the limit process, the distances between all points have to be of the same order.

These examples strongly suggest to think of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ as the *space of all polynomial curves of degree* $n-1$ *passing through the origin*, or better a *generic point* of Hilb_0^n can be seen as *the* $(n-1)$*-jet of a curve passing through the origin.* This clearly generalizes the projectivization which we get for $n = 2$. It can be shown that collapsing $(kt, \gamma(kt))$ for $k = 1, ..., n$ with $\gamma(t) = a_1 t + a_2 t^2 + ... + a_n t^n$ to $(0, 0)$ leads to the ideal

$$\left\langle x^{n+1}, -y + a_1 x + a_2 x^2 + ... + a_n x^n \right\rangle .$$

3.2 Blowup of configuration spaces

Another way to look on the punctual Hilbert scheme is its relation to the configuration space $\mathrm{Sym}^n(\mathbb{C}^2)$ of (not necessarily distinct) n points of the plane. More precisely, $\mathrm{Sym}^n(\mathbb{C}^2)$ is defined as the quotient of $(\mathbb{C}^2)^n$ by the permutation group $\mathcal{S}_n$ acting by permuting the points. This space is singular since the symmetric group does not act freely. The Hilbert scheme is a blowup, i.e. a minimal resolution, of the configuration space. Roughly speaking, when two or more points coincides, the Hilbert scheme gives an extra information "how they collide".

To any ideal I of codimension n, one can associate its support, i.e. the algebraic variety defined by I. Taking multiplicities into account (defined by localization), the support of I consists of n points. Since the order of these points does not matter, they can be seen as an element of $\mathrm{Sym}^n(\mathbb{C}^2)$. This map

$$\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$$

is called the **Chow morphism**.

It is clear that the Chow morphism is an isomorphism between the set of subschemes without multiplicities and the non-singular points of $\mathrm{Sym}^n(\mathbb{C}^2)$. In fact, one can show that the Chow morphism is birational. Thus, we get

$$\dim \mathrm{Hilb}^n(\mathbb{C}^2) = 2n.$$

Theorem 6 (Grothendieck, Fogarty)**.** *The space* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *is a smooth and irreducible scheme. It is a blowup of the configuration space* $\mathrm{Sym}^n(\mathbb{C}^2)$.

The original paper of Fogarty is [11]. An accessible proof using combinatorics is given in Haiman [15]. We just give some ideas and invite the reader to look up the details in Haiman's paper. It is interesting to notice that the theorem only holds in dimension 2 since $\mathrm{Hilb}^n(\mathbb{C}^m)$ is neither irreducible nor smooth in general.

Let's start by giving explicit coordinate charts. This can be done in terms of Young diagrams. A **Young diagram** D is a finite subset of $\mathbb{N} \times \mathbb{N}$ such that whenever $(i, j) \in D$ then the rectangle defined by (i, j) and $(0, 0)$ is entirely in D. Usually, one uses matrix-like notations such that $(0, 0)$ is in the upper left corner. Figure 3 gives examples of Young diagrams (ignoring the entries for the moment).

1	y	y^2
x	xy	
x^2	x^2y	
x^3		

1	y	y^2	y^3
x	xy		
x^2			
x^3			

Fig. 3: Examples of Young diagrams

These diagrams play an important role for visualizing partitions. The set of all Young diagrams with n squares is in bijection with the partitions of n. Indeed, given a Young diagram, you can read off the partition by adding the lines. Figure 3 for example gives 8=3+2+2+1 and 8=4+2+1+1.

Now, to any Young diagram D, we can associate

$$\mathcal{B}_D = \{x^i y^j \mid (j, i) \in D\}$$

a subset of the standard basis of the polynomial ring $\mathbb{C}[x, y]$ (see again figure 3). We then set

$$U_D = \{I \in \mathrm{Hilb}^n(\mathbb{C}^2) \mid \mathcal{B}_D \text{ spans } \mathbb{C}[x, y]/I\}.$$

On U_D, we can decompose any monomial in the basis $\mathcal{B}_D$:

$$x^r y^s = \sum_{(j,i)\in D} c_{ij}^{rs} x^i y^j \mod I$$

Proposition 1. *These U_D are open affine subvarieties covering* Hilb^n. *The coordinate ring $\mathcal{O}_{U_D}$ is generated by the c_{ij}^{rs} with $(j,i) \in D$. That gives the scheme structure of* Hilb^n.

See [15], proposition 2.1, for the complete proof. We only give a nice proof that the U_D cover all of Hilb^n.

Proof that all of Hilb^n *is covered.* Let I be any ideal of codimension n. We want to show that there is a basis of $\mathbb{C}[x,y]/I$ which is of the form $\mathcal{B}_D$ for some Young diagram D. We describe an algorithm giving a basis of this form:
Since $1 \notin I$, we can take 1 as the first basis vector of $\mathbb{C}[x,y]/I$. Then, you run through $[0,n] \times [0,n]$ by columns, starting at $(0,0)$. Every time the vector $x^i y^j$ is linearly independent from those visited before, select it as an element for our basis (see figure 4).

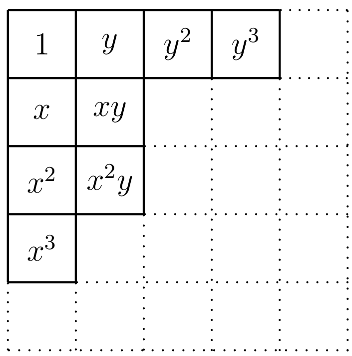

Fig. 4: Getting a Young diagram

In this way, we get a set of linearly independent vectors. We have to show that there are n of them and that they form a Young diagram.
Let's start with the last one. If the set D of selected squares does not form a Young diagram then there is a selected square (j,i) such that $(j,i-1)$ or $(j-1,i)$ does not belong to D. If $(j,i-1)$ does not belong to D, then it is a linear combination of the previous squares. Since $x^i y^j = x(x^{i-1}y^j)$ and since multiplication by x corresponds to a *vertical shift*, we get $x^i y^j$ as a linear combination of the previous squares, a contradiction. The same arguments holds for $(j-1,i)$ with a *horizontal shift* given by multiplication by y.
Finally, if we had not selected n squares, there is a vector v linear independent to D. Since v is a sum of monomials, there is at least one square (j,i) in $\mathbb{N} \times \mathbb{N}$ which is linearly independent to D. The argument above shows that the set of squares of the rectangle defined by $(0,0)$ and (j,i) is a free set. So we have $ij \leq n$ which implies $i \leq n$ and $j \leq n$. So the square (j,i) was already selected in our process. □

This explicit construction by "running in columns" gives a visualization of ideals of finite codimension. In the discrete plane $\mathbb{N} \times \mathbb{N}$, there are marked squares (with a star $\star$, see figure 4), one for each column, corresponding to a relation in I such that all squares coming before the star in their column form a Young diagram for I.

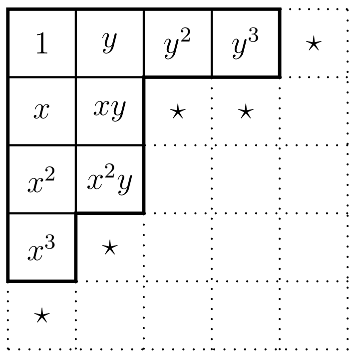

Fig. 5: Ideal of the Hilbert scheme

We can ask which relations in the marked squares can be obtained by the process "running in columns". For example you can obtain $\langle x^2, xy, y^2 - ax \rangle$. But it is impossible to get $\langle x^3, xy - ay - bx - cx^2, y^2 \rangle$ with $a \neq 0$ because these relations imply that $x^2 y \in I$ and then that $y + \frac{b}{a}x + \frac{c}{a}x^2 \in I$ which contradicts the freeness of (x, x^2, y).

First of all, it is sufficient to keep only the first star in each row (when there are at least two stars ín a row, like the second row in figure 5). Indeed, if there are two neighboring stars in squares (j, i) and $(j+1, i)$, the relation of $x^i y^{j+1}$ in terms of elements of D obtained by the "running in columns" is the relation of $x^i y^j$ multiplied by y. Otherwise we would get a relation among elements of D which is a free set.

When you keep only the first star in each row, the relations you can obtain are *reduced Gröbner bases with respect to the monomial order $x < y$*. So we will say a few words about Gröbner bases, see [9] for a detailed account.

A **Gröbner basis** G of an ideal I in a polynomial ring $\mathbb{C}[x_1, ..., x_n]$ with respect to some monomial order is a generating set of I such that the set of leading monomials of G generates all leading monomials in I (which form an ideal). Note that we need a total monomial order to define the leading monomial. A Gröbner basis G is said to be **reduced** if all leading coefficients in G are 1 and if every monomial of a term in G is not in the ideal generated by the leading monomials of the other elements of G. A fundamental theorem states the *existence and uniqueness of reduced Gröbner bases.*

The process "running in columns" corresponds to the monomial order $x < y$ (lexicographic order). The marked squares correspond to the leading monomial of the relation associated to the square since this relation links the square to the squares visited before.

Proposition 2. *Keeping only the first relation in each row obtained by the process described in the previous proof, we get a reduced Gröbner bases. Conversely, every reduced Gröbner basis generating an ideal of finite codimension can be obtained in this way.*

Proof. For the first part, we have already justified that the first stars in each row form a generating set. Whenever a polynomial P is in I its leading monomial cannot be in D, otherwise we would get a relation in D which is a free set modulo I. By the definition of the marked squares, we see that the leading monomial of P is divisible by a leading monomial of a marked square. So we have a Gröbner basis. Since we kept only one star in each row and since a relation links a marked square to squares in D, it is clear that this Gröbner basis is reduced.
The converse follows from the uniqueness of Gröbner bases and the first part of the proof. □

Notice that starting with a generating set of I which does not form a Gröbner basis, we can apply **Buchberger's algorithm** to obtain the reduced Gröbner basis. Notice further that running through $\mathbb{N} \times \mathbb{N}$ in rows gives reduced Gröbner bases with respect to the monomial order $x > y$.

3.3 Commuting matrices

Interestingly, there is a description of the punctual Hilbert scheme in terms of linear algebra.

For a given ideal I of codimension n, we can consider the multiplication by x as a linear operator A of the n-dimensional space $\mathbb{C}[x,y]/I$. In the same way, we can consider the multiplication by y as a linear operator B. Since these two operations commute, the operators A and B do so too. In addition, the element $1 \in \mathbb{C}[x,y]/I$ is a cyclic vector for A and B, meaning that $\mathbb{C}[x,y]/I$ is generated by $\{A^n B^m.1 \mid n,m \in \mathbb{N}\}$. We consider A and B as conjugacy classes of matrices.

Proposition 3. *The punctual Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *is in bijection with conjugacy classes of matrices* $A, B \in \mathcal{M}_n(\mathbb{C})$ *which commute and admit a cyclic vector.*

Proof. We will give the inverse construction: given two commuting matrices A and B which admit a cyclic vector, put

$$I = \{P \in \mathbb{C}[x,y] \mid P(A,B) = 0\}.$$

This defines clearly an ideal. The fact that A and B commute implies that I is of codimension at most n and the existence of a cyclic vector shows that the codimension is at least n. It is easy to check that this gives the inverse construction to the above one. $\square$

We can give a description of the Chow morphism in this picture: Since A and B commute, it is possible to put them simultaneously in upper triangular form with diagonal entries $(\lambda_1, ..., \lambda_n)$ for A and $(\mu_1, ..., \mu_n)$ for B. Then the Chow morphism $\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$ is given by

$$\pi(A,B) = ((\lambda_1, \mu_1), ..., (\lambda_n, \mu_n)) \tag{5}$$

To prove this, first suppose that all diagonal entries are different: $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Then, the matrices A and B are simultaneously diagonalizable. In that case, the set of polynomials P with $P(A,B) = 0$ coincides with the set of polynomials vanishing on the points (λ_i, μ_i) for $1 \leq i \leq n$. Hence, in that case the Chow morphism is given by the map defined in (5). We conclude by density of diagonalizable matrices.
Therefore, the zero-fiber $\mathrm{Hilb}_0^n(\mathbb{C}^2) = \pi^{-1}(0)$ is given by *nilpotent commuting matrices admitting a cyclic vector.*

An application of this viewpoint is the irreducibility of the set of all commuting nilpotent matrices. See [4] for details.

3.4 Zero-fiber of the Hilbert scheme

In this subsection, we investigate more in detail the zero-fiber of the Hilbert scheme since it will show up in the definition of the higher complex structure. In particular, we will give its dimension.

We think of an ideal I in Hilb_0^n as generated by marked squares around a Young diagram as shown in figure 5 above. Since I is supported on 0, no relation has constant terms. Furthermore, we saw in the previous section that the zero-fiber corresponds to commuting nilpotent matrices. Thus, we get $\langle x, y\rangle^n \subset I$, which means that $x^k y^{n-k} = 0$ in I for $k = 0, ..., n$. This allows to *"compute formally" in* Hilb_0^n *pushing problems in higher and higher orders such that they will disappear* *at order n*. We will use this argument several times in the sequel where it will be clearer what it means.

Theorem 7. *The zero-fiber* $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ *is a irreducible scheme of dimension* $n-1$.

A proof can be found in [4]. Notice that unlike the Hilbert scheme, the zero-fiber is not smooth. We will give a simple argument to compute the dimension of Hilb_0^n.

Proof of the dimension. In the previous section, we saw that Hilb_0^n is the set of commuting nilpotent matrices A and B admitting a cyclic vector up to conjugacy. A generic nilpotent matrix can be put into Jordan normal form with only one Jordan block (the set of these matrices is dense in the set of nilpotent matrices). So to compute the dimension at a generic point of Hilb_0^n, we compute the dimension of the centralizer of a Jordan block.

A direct computation gives that a matrix B commutes with a Jordan block A iff it is upper triangular with equal entries in each over-diagonal (line parallel to the main diagonal situated above it). Since B is nilpotent, there are only zeros on the diagonal.

Therefore, there are $n-1$ degrees of freedom for B. The extra condition that A and B admit a cyclic vector can only decrease this estimate. Finally we see that the dimension of Hilb_0^n at a generic point is at least $n-1$ by giving an explicit example of an ideal with $n-1$ degrees of freedom (see below). □

We can see that the special elements of Hilb_0^n

$$\left\langle x^n, -y + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} \right\rangle$$

and

$$\left\langle y^n, -x + b_1 y + b_2 y^2 + ... + b_{n-1} y^{n-1} \right\rangle$$

have $n-1$ degrees of freedom. In fact, these are the *generic ideals* in the sense that the set of all ideals of Hilb_0^n which are not of this form have dimension strictly smaller than $n-1$. See [18] (corollary 1) for a proof of that fact. Notice that the generic ideals corresponds to Young diagrams which are either a single column or a single row.

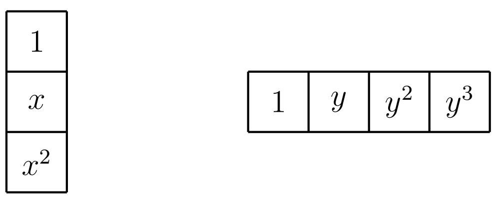

Fig. 6: Young diagrams of generic ideals

3.5 Interlude: Series reversion*

We just mentioned that the generic ideal in $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is of the form

$$\left\langle x^n, -y + a_1 x + a_2 x^2 + ... + a_{n-1}x^{n-1}\right\rangle$$

or

$$\left\langle y^n, -x + b_1 y + b_2 y^2 + ... + b_{n-1}y^{n-1}\right\rangle$$

or both. So it is natural to ask about the coordinate change in the case when an ideal can be expressed in both forms, i.e. we are looking for an expression of the b_i in terms of the a_j.

The algebraic approach is to insert the equation $x = b_1 y + b_2 y^2 + ... + b_{n-1}y^{n-1}$ into $y = a_1 x + a_2 x^2 + ... + a_{n-1}x^{n-1}$ and to compare coefficients. One gets for the first values:

$$\begin{aligned}
b_1 =& \frac{1}{a_1}\\
b_2 =& \frac{-a_2}{a_1^3}\\
b_3 =& \frac{1}{a_1^5}(-a_1 a_3 + \mathbf{2} a_2^2)\\
b_4 =& \frac{1}{a_1^7}(-a_1^2 a_4 + 5a_1 a_2 a_3 - \mathbf{5} a_2^3)\\
b_5 =& \frac{1}{a_1^9}(-a_1^3 a_5 + 6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 - 21 a_1 a_2^2 a_3 + \mathbf{14} a_2^4)
\end{aligned}$$

The last coefficient seems to be a **Catalan number** with alternating sign which triggered my interest (Haiman's paper is about a generalization of Catalan numbers). Other striking facts: in the expression of b_n, the number of a_i's is $n-1$ and the sum of their indices (with multiplicity) is equal to $2n-2$. That is, the terms are of the form $a_1^{i_1}...a_n^{i_n}$ with

$$i_1 + i_2 + ... + i_n = n-1 \text{ and } i_1 + 2i_2 + ... + ni_n = 2n-2.$$

This is equivalent to $i_1 = n - i_2 - ... - i_n$ and $i_2 + 2i_3 + ... + (n-1)i_n = n-1$. The last equation gives a bijection to partitions of $n-1$: for a given partition of $n-1$, if i_k denotes the number of terms $k-1$ in the partition, we get $i_2 + 2i_3 + ... + (n-1)i_n = n-1$.

Fixing some notation: we will write "$\nu \vdash n$" for "*ν is a partition of n*", $\nu = (\nu_1, ..., \nu_n)$ with ν_k the *number of k's* in the partition and $|\nu| = \nu_1 + ... + \nu_n$. In the Young diagram associated to ν, the number ν_k counts the number of lines of length

k and $|\nu|$ is the number of lines.

So we conjecture that

$$b_n \stackrel{?}{=} \frac{1}{a_1^{2n-1}} \sum_{\nu \vdash (n-1)} c_\nu a_1^{n-1-|\nu|} a_2^{\nu_1} ... a_n^{\nu_{n-1}} \tag{6}$$

with some constants c_ν. Analyzing some numerical data, we further conjecture an explicit formula for these constants:

$$c_\nu \stackrel{?}{=} \frac{(-1)^{|\nu|}(|\nu| + n - 1)!}{n!\nu_1!...\nu_{n-1}!} \tag{7}$$

For the coefficient in the last term of b_n, we have $\nu = (0, n-1, 0..., 0)$, so we get $(-1)^{n-1}\frac{1}{n}\binom{2n-2}{n-1}$, a Catalan number with sign. The other coefficients seem to have no link to Haiman's t, q-Catalan numbers.

The algebraic approach gives a recursive formula with several partitions involved. This formula can be used to prove the conjectured form (6) but does not give (7) (at least not easily). I think that one needs a combinatorial interpretation to prove (7) with this approach. I only succeeded to prove the appearance of the Catalan numbers with a combinatorial argument.

In order to prove (7) we give an analytic approach. In fact, our problem is that of the *formal reversion of power series*, i.e. given $y = f(x) = a_1 x + a_2 x^2 + ...$ a formal power series, we put $f^{-1}(y) = b_1 x + b_2 x^2 + ...$ the power series expansion of the inverse function (with respect to composition, not multiplication) and we want to express at least formally the b_i in terms of the a_j. Note that in the case of the Hilbert scheme, this formal correspondence will be exact since we cut the series expansions at level n.

Now, we can express the coefficients by derivatives: $k!a_k = f^{(k)}(0)$ and $k!b_k = (f^{-1})^{(k)}(0)$. All we have to do is to express the derivatives of f^{-1} in terms of derivatives of f. The first derivatives are given by

$$\begin{aligned}
(f^{-1})' =& \frac{1}{f' \circ f^{-1}} \\
(f^{-1})'' =& \frac{-f'' \circ f^{-1}}{(f' \circ f^{-1})^3} \\
(f^{-1})''' =& \frac{1}{(f' \circ f^{-1})^5}(-(f' \circ f^{-1})(f''' \circ f^{-1}) + 3(f'' \circ f^{-1})^2) \\
(f^{-1})^{(4)} =& \frac{1}{(f')^7}(-(f')^2 f^{(4)} + 10 f' f'' f''' - 15(f'')^3)
\end{aligned}$$

where we omitted the "$\circ f^{-1}$" in the last line. This is quite similar to the formulas for the b_i. So we conjecture that there are coefficients ε_ν such that

$$(f^{-1})^{(n)} \overset{?}{=} \frac{1}{(f')^{2n-1}} \sum_{\nu \vdash (n-1)} \varepsilon_\nu (f')^{n-1-|\nu|} (f'')^{\nu_1} ... (f^{(n)})^{\nu_{n-1}} \tag{8}$$

Equation (7) and (8) are of course related since $k! b_k = (f^{-1})^{(k)}(0)$. Thus, from our conjecture on c_ν, we get a conjecture on the coefficients ε_ν which we will be able to prove:

$$\varepsilon_\nu \overset{?}{=} \frac{(-1)^{|\nu|}(|\nu| + n - 1)!}{(2!)^{\nu_1} ... (n!)^{\nu_{n-1}} \nu_1! ... \nu_{n-1}!} \tag{9}$$

By induction on ν and n, we get by omitting terms $\circ f^{-1}$:

$$\begin{aligned}
(f^{-1})^{(n+1)} = & \left(\frac{1}{(f')^{2n-1}} \sum_{\nu \vdash (n-1)} \varepsilon_\nu (f')^{n-1-|\nu|} (f'')^{\nu_1} ... (f^{(n)})^{\nu_{n-1}} \right)' \\
= & \frac{1}{(f')^{2n+1}} \left(\sum_{\nu \vdash (n-1)} \varepsilon_\nu (-n - |\nu|)(f')^{n-1-|\nu|} (f'')^{\nu_1+1} (f''')^{\nu_2} ... (f^{(n)})^{\nu_{n-1}} \right. \\
& \left. + \sum_{l=1}^{n-1} \sum_{\nu \vdash (n-1)} \varepsilon_\nu \nu_l (f')^{n-|\nu|} (f'')^{\nu_1} ... (f^{(l+1)})^{\nu_l - 1} (f^{(l+2)})^{\nu_{l+1}+1} ... (f^{(n)})^{\nu_{n-1}} \right) \\
= & \frac{1}{(f')^{2n+1}} \sum_{\tilde{\nu} \vdash n} \varepsilon_{\tilde{\nu}} (f')^{n-|\tilde{\nu}|} (f'')^{\tilde{\nu}_1} ... (f^{(n+1)})^{\tilde{\nu}_n}
\end{aligned}$$

where for $\tilde{\nu} \vdash n$ we put

$$\varepsilon_{\tilde{\nu}} = (-n - |\tilde{\nu}| + 1) \varepsilon_{(\tilde{\nu}_1 - 1, \tilde{\nu}_2, ..., \tilde{\nu}_n)} + \sum_{l=1}^{n-1} (\tilde{\nu}_l + 1) \varepsilon_{(\tilde{\nu}_1, ... \tilde{\nu}_l + 1, \tilde{\nu}_{l+1} - 1, ..., \tilde{\nu}_n)}$$

By induction hypothesis, we can insert (9) in the last expression which yields

$$\begin{aligned}
\varepsilon_{\tilde{\nu}} = & \frac{(-1)^{|\tilde{\nu}|}(|\tilde{\nu}| + n - 1)!}{(2!)^{\tilde{\nu}_1} ... ((n+1)!)^{\tilde{\nu}_n} \tilde{\nu}_1! ... \tilde{\nu}_n!} \left(2! \tilde{\nu}_1 + \sum_{l=1}^{n-1} (\tilde{\nu}_l + 1) \frac{(l+2)! \tilde{\nu}_{l+1}}{(l+1)!(\tilde{\nu}_l + 1)} \right) \\
= & \frac{(-1)^{|\tilde{\nu}|}(|\tilde{\nu}| + n)!}{(2!)^{\tilde{\nu}_1} ... ((n+1)!)^{\tilde{\nu}_n} \tilde{\nu}_1! ... \tilde{\nu}_n!}
\end{aligned}$$

since $2\tilde{\nu}_1 + 3\tilde{\nu}_2 + ... + (n+1)\tilde{\nu}_n = |\tilde{\nu}| + n$. This is exactly the conjectured form (9). Therefore, we succeeded to show our four conjectures (6) to (9).
Since the procedure of derivation gives only integers, we get as a corollary that

$$\varepsilon_\nu = \frac{(-1)^{|\nu|}(|\nu| + n - 1)!}{(2!)^{\nu_1} ... (n!)^{\nu_{n-1}} \nu_1! ... \nu_{n-1}!} \in \mathbb{Z} \text{ for } \nu \vdash n - 1$$

This can also be seen by a combinatorial argument: if ν is a partition of $n-1$, then

$$\frac{(n-1)!}{\prod \nu_j!(j!)^{\nu_j}}$$

is an integer because it counts the number of set-theoretic partitions of $\{1, 2, ..., n-1\}$ with ν_k sets of cardinal k for all k (imagine any permutation, the first ν_1 elements will be one-element subsets, the next ν_2 pairs will be the two-element subsets etc.). By adding one element to each subset, we get a bijection to partitions of $\{1, ..., n-1+|\nu|\}$ with subsets of at least two elements. Now, ν_k counts the number of $(k+1)$-element subsets. Thus

$$\frac{(n-1+|\nu|)!}{\prod \nu_j!(j+1)!^{\nu_j}} \in \mathbb{N}$$

More surprising is the fact that also c_ν is an integer which can be easily seen in the algebraic approach. Hence

$$\frac{(n-1+|\nu|)!}{n!\prod \nu_j!} \in \mathbb{N} \text{ for } \nu \vdash n-1.$$

This generalizes the property of Catalan numbers, that $\binom{2n}{n}$ is divisible by $n+1$.

These formulas are already known: the reversion of power series is known as **Lagrange's inversion theorem**. The identities (8) and (9) inverses **Faà di Bruno's formula** which give an expression for derivatives of a composition $(f \circ g)^{(n)}$. See for instance [23], page 411 to 413 for the reversion of power series and [19] for Faà di Bruno's formula.

3.6 Orthogonal viewpoint*

In this subsection, we describe a pairing on $\mathbb{C}[x, y]$ which will allow a description of the zero-fiber of the Hilbert scheme as the space of translation-invariant finite-dimensional vector subspaces of $\mathbb{C}[x, y]$. Every vector space is over $\mathbb{C}$ in this part.

Let's start with the definition of the pairing:

$$\langle P, Q\rangle := P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}).Q\Big|_{x=y=0}$$

where the little point means "applied to".
To see what is going on and why this formula gives a pairing, let's compute its value in the standard basis $\{x^n y^m \mid n, m \in \mathbb{N}\}$. We easily get

$$\left\langle x^n y^m, x^{n'} y^{m'}\right\rangle = n!m!\delta_{n,n'}\delta_{m,m'}$$

Thus, we see that the pairing is nothing else than the standard inner product of $\mathbb{R}[x,y]$ with weights $n!m!$ for $x^n y^m$ extended by $\mathbb{C}$-bilinearity. This shows in particular that $\langle .,. \rangle$ is symmetric and non-degenerate.

Once we have a pairing, we can define the orthogonal complement $S^\perp$ of any subset S of $\mathbb{C}[x,y]$. In the case where S is an ideal, its orthogonal has special properties:

Proposition 4. *Let I be an ideal of $\mathbb{C}[x,y]$. Then $I^\perp$ is a vector space stable under derivation and translation.*

Proof. For any subset S, it is easy to check that $S^\perp$ is a vector space, using the $\mathbb{C}$-bilinearity of the pairing. For the invariance, notice the following fundamental identity:

$$\langle PQ, R\rangle = \left\langle P, Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}).R \right\rangle \tag{10}$$

Thus, if P is an element of I, Q any polynomial and R in $I^\perp$, we get that $Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}).R$ also belongs to $I^\perp$. Therefore $I^\perp$ is stable under derivation. Finally, since

$$P(x+a, y+b) = \exp(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}).P(x,y)$$

we see that $I^\perp$ is also invariant under all translations. □

Remark. *The invariance of $I^\perp$ under translation shows that $I^\perp$ is a* ***subcoalgebra*** *of $\mathbb{C}[x,y]$ in the following sense: If $P \in I^\perp$, we have $\Delta P \in I^\perp \otimes I^\perp$ where $\Delta P(x_1,y_1,x_2,y_2) = P(x_1+x_2, y_1+y_2)$ is the dual operation to addition.*

Now, we can describe explicitly the orthogonal of the zero-fiber of the punctual Hilbert scheme, defined by taking the orthogonal to every ideal $I \in \mathrm{Hilb}_0^n$:

Proposition 5. *The orthogonal of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is the space of all vector subspaces of $\mathbb{C}[x,y]$ of dimension n which are invariant under translations. The same holds true when you replace "translation" by "derivation".*

Proof. The orthogonal complement sends vector spaces of codimension n to vector spaces of dimension at most n. In fact, if we work in the ring of formal power series $\mathbb{C}[[x,y]]$ then the orthogonal is of dimension exactly n. But for the zero-fiber Hilb_0^n, we cut at level n, that is $\langle x,y\rangle^n = 0$. Thus, for $I \in \mathrm{Hilb}_0^n(\mathbb{C}^2)$, we see that $I^\perp$ is of dimension n and by the previous proposition is invariant under translations and derivations. Conversely, if J is a n-dimensional vector space invariant under all translations, it is especially invariant under all derivations (=infinitesimal translations). Then formula (10) shows that $J^\perp$ is an ideal. Finally, since J is finite-dimensional, there is an integer m such that $\langle x,y\rangle^m \subset J^\perp$ showing that $J^\perp$ is supported on 0. $\square$

Since $\langle .,.\rangle$ is an inner product, we can identify the Hilbert scheme with the space of translation-invariant finite-dimensional subspaces.

Let's give some examples:
For $I = \langle x^2, xy, y^2\rangle$, we get $I^\perp = \mathrm{Vect}(1, x, y)$.
For $I_2 = \langle x^2, -y + a_1x\rangle$, we get $I_2^\perp = \mathrm{Vect}(1, x + a_1y)$.
For $I_3 = \langle x^3, -y + a_1x + a_2x^2\rangle$, we get $I_3^\perp = \mathrm{Vect}(1, x + a_1y, (x + a_1y)^2 + 2a_2y)$.
For $I_4 = \langle x^4, -y + a_1x + a_2x^2 + a_3x^3\rangle$, we get

$$I_4^\perp = \mathrm{Vect}(1, x + a_1y, (x + a_1y)^2 + 2a_2y, (x + a_1y)^3 + 6a_2y(x + a_1y) + 6a_3y)$$

For the interested reader, we indicate how we computed the complement because it is another example of the "formal computation" in Hilb_0^n. In the case of I_3 for instance, we try to get a term in $I_3^\perp$ starting with x^2. To be orthogonal to $-y+a_1x+a_2x^2$, we have to add $2a_2y$ and for $x(-y+a_1x+a_2x^2)$ we get $2a_1xy$. Finally orthogonality with $y(-y + a_1x + a_2x^2)$ gives a term $a_1^2y^2$. Thus, we get $x^2 + 2a_1xy + a_1^2y^2 + 2a_2y \in I_3^\perp$. The process stops because $\langle x,y\rangle^3 = 0 \mod I_3$.

We finish this subsection by an explicit formula for the dual of I_n: Adopting the notations for partitions as in the previous subsection, we get for $I_{n+1} = \langle x^{n+1}, -y + a_1x + a_2x^2 + ... + a_nx^n\rangle$ that

Proposition 6.

$$\begin{aligned} I_{n+1}^\perp =& I_n^\perp \oplus \mathrm{Vect}(\sum_{l=0}^{n} \sum_{\substack{\nu \vdash n-l \\ \nu_1=0}} \frac{n!}{l!\nu_2!...\nu_n!} a_2^{\nu_2}...a_n^{\nu_n} y^{|\nu|}(x + a_1y)^l) \\ =& I_n^\perp \oplus \mathrm{Vect}(\sum_{m=0}^{n} \sum_{\nu \vdash n-m} \frac{n!}{m!\nu_1!\nu_2!...\nu_n!} a_1^{\nu_1}a_2^{\nu_2}...a_n^{\nu_n} y^{|\nu|}x^m) \end{aligned}$$

Proof. The formula is true for $n = 1$, so we use induction. To prove the formula, it is clear that $I_n^\perp \subset I_{n+1}^\perp$ since $I_{n+1} \subset I_n$. So it is sufficient to find a vector orthogonal to I_{n+1} which is linear independent to $I_n^\perp$. The candidate $P(x,y)$ for this vector which is given in the formula above is clearly linear independent from $I_n^\perp$ since its degree in x is higher than that of any element of $I_n^\perp$ (by induction hypothesis). In addition, it is annulated by $\frac{\partial^{n+1}}{\partial x^{n+1}}$ since it has degree n in x. We will show that

$$(-\frac{\partial}{\partial y} + a_1\frac{\partial}{\partial x} + ... + a_n\frac{\partial^n}{\partial x^n}).P(x,y) = 0.$$

Since any element of I_{n+1} is of the form $T = Ax^{n+1} + B(-y + a_1 x + ... + a_n x^n)$, we get

$$\langle T, P\rangle = \left\langle A, \frac{\partial^{n+1} P}{\partial x^{n+1}}\right\rangle + \left\langle B, (-\frac{\partial}{\partial y} + a_1 \frac{\partial}{\partial x} + ... + a_n \frac{\partial^n}{\partial x^n}).P\right\rangle = 0$$

which shows that $P \in I_{n+1}^{\perp}$.

We first compute the derivative of P with respect to x:

$$\begin{aligned}\frac{\partial P}{\partial x} =& \sum_{l=0}^{n} \sum_{\substack{\nu \vdash n-l \\ \nu_1 = 0}} \frac{n!}{(l-1)! \nu_2! ... \nu_n!} a_2^{\nu_2} ... a_n^{\nu_n} y^{|\nu|} (x + a_1 y)^{l-1} \\ =& n \sum_{l=0}^{n-1} \sum_{\substack{\nu \vdash n-l-1 \\ \nu_1 = 0}} \frac{(n-1)!}{l! \nu_2! ... \nu_n!} a_2^{\nu_2} ... a_n^{\nu_n} y^{|\nu|} (x + a_1 y)^{l}\end{aligned}$$

To finish, let's compute the y-derivative of P:

$$\begin{aligned}\frac{\partial P}{\partial y} =& \sum_{l=0}^{n} \sum_{\substack{\nu \vdash n-l \\ \nu_1 = 0}} \frac{n!}{(l-1)! \nu_2! ... \nu_n!} a_2^{\nu_2} ... a_n^{\nu_n} y^{|\nu|-1} (x + a_1 y)^{l-1} (a_1 y + |\nu|(x + a_1 y)) \\ =& \sum_{m=0}^{n-1} \frac{n!}{m!} a_{n-m} \sum_{l=0}^{m} \sum_{\substack{\nu \vdash m-l \\ \nu_1 = 0}} \frac{m!}{l! \nu_2! ... \nu_n!} a_2^{\nu_2} ... a_n^{\nu_n} y^{|\nu|} (x + a_1 y)^{l} \\ =& \sum_{m=0}^{n-1} a_{n-m} \frac{\partial^{n-m}}{\partial x^{n-m}}.P\end{aligned}$$

where we get from the first to the second line by using the fact that to a partition $\nu \vdash n - l$, we can associate $|\nu|$ partitions of smaller numbers by forgetting one of the terms and we get from the second to the third line by using our computation of the x-derivative of P, iterated several times. □

4 Higher complex structure

In this final part, we define the higher complex structure and explore its main properties. For complex structures, we were interested in them up to diffeomorphisms isotopic to the identity. We will see that for higher complex structures, it is better to enlarge the group of diffeomorphisms. We then explore the local and global theory of that new structure.

4.1 Definition and basic properties

In section 2, we saw that a complex structure on a surface Σ is uniquely given by a section σ of $\mathbb{P}(T^{*\mathbb{C}}\Sigma)$, the (pointwise) projectivized complexified cotangent space, such that at any point $z \in \Sigma$, $\sigma(z)$ and $\bar{\sigma}(z)$ are linearly independent. In the previous section, we saw that the projectivization is a special case of the zero-fiber of the punctual Hilbert scheme for $n = 2$:

$$\mathbb{P}(T^{*\mathbb{C}}\Sigma) = \mathrm{Hilb}^2_0(T^{*\mathbb{C}}\Sigma).$$

Presented in this manner, it is easy to guess our definition for the higher complex structure. Note that a chart z on Σ gives vectors $p = \frac{\partial}{\partial z}$ and $\bar{p} = \frac{\partial}{\partial \bar{z}}$ which can be seen as linear functionals, i.e. linear coordinates, on $T^{*\mathbb{C}}\Sigma$.

Definition 5. *A* ***higher complex structure*** *of order n on a surface Σ, for short n-**complex structure**, is a section of $\mathrm{Hilb}^n_0(T^{*\mathbb{C}}\Sigma)$ such that at each point z we have $I(z) \oplus \bar{I}(z) = \langle p, \bar{p}\rangle$.*

For $n = 2$, the extra condition $I \oplus \bar{I} = \langle p, \bar{p}\rangle$ simply reads $\mu_2\bar{\mu}_2 \neq 1$ which is exactly what we had for the complex structure (see 2.3). So we recover the complex structure for $n = 2$. We chose the name of "higher complex structure" because we hope to show a strong relation to higher Teichmüller spaces.

Any higher complex structure gives in particular a complex structure by forgetting all μ_k apart from μ_2. Hence, a higher complex structure gives an orientation on the surface. We say that a higher complex structure is **compatible** if this induced orientation coincides with the surface orientation.

As for the almost complex structure, we will analyze the n-complex structure in three steps. We start with the analysis at one point. The condition $I \oplus \bar{I} = \langle p, \bar{p}\rangle$ gives the form of the ideal at each point:

Proposition 7. *A compatible n-complex structure at any point z is given by an ideal of the form*

$$I(z) = \left\langle p^n, -\bar{p} + \mu_2(z,\bar{z})p + ... + \mu_n(z,\bar{z})p^{n-1} \right\rangle \text{ with } |\mu_2| < 1.$$

The coefficients μ_k are called ***higher Beltrami coefficients****.*

Proof. Let I_1 be the set of all degree 1 polynomials which appear in an element of I. It is clear that I_1 is a vector subspace of $\mathbb{C}^2$ since I is a vector space. We will show that I_1 is of dimension 1.
If $I_1 = \{0\}$, then so is $\bar{I}_1 = \{0\}$. But by $I \oplus \bar{I} = \langle p, \bar{p} \rangle$, we get $I_1 \oplus \bar{I}_1 = \mathbb{C}^2$ which is absurd. If $I_1 = \mathbb{C}^2$ then $I = \langle p, \bar{p} \rangle$ which contradicts the fact that it is of codimension $n \geq 2$. Indeed, take any polynomial P without constant term. Since we assume $I_1 = \mathbb{C}^2$, we can eliminate the homogenous part of degree 1 of P, introducing only terms of higher degree. By multiplying elements of I_1 by p or $\bar{p}$, we can also eliminate all terms of degree 2 in P introducing only terms of degree at least 3 and so on. Since we have $p^k\bar{p}^{n-k} = 0 \mod I$ for all k, this process will stop and P will be in I.
Therefore $I_1 = \text{Vect}(ap + b\bar{p})$ is of dimension 1. So $\bar{I}_1 = \text{Vect}(\bar{a}\bar{p} + \bar{b}p)$ and the condition $I \oplus \bar{I} = \langle p, \bar{p} \rangle$ is equivalent to $a\bar{a} \neq b\bar{b}$. Since the n-complex structure is compatible, we have $|\mu_2| = |a/b| < 1$. In particular, $b \neq 0$ which gives $I_1 = \text{Vect}(-\bar{p} + \mu_2 p)$.
Finally, since $-\bar{p} + \mu_2 p \in I_1$, there is a relation of the form $\bar{p} = \mu_2 p +$ higher terms in I. Iterating this equality by replacing it in any $\bar{p}$ appearing in the higher terms, we will get an expression of $\bar{p}$ in terms of monomials in p (this procedure will stop). Since $p^n = 0$ in I, we get

$$\bar{p} = \mu_2 p + \mu_3 p^2 + ... + \mu_n p^{n-1} \mod I.$$

To give an example, we get for $n = 4$ and $\bar{p} = ap + bp\bar{p}$ that

$$\bar{p} = ap + bp(ap + b(ap + b\bar{p})) = ap + abp^2 + ab^2p^3.$$

□

Remark. *In the proof, we see that if a n-complex structure given by $I(z)$ is not compatible, then the conjugated structure $\bar{I}(z)$ is compatible. Hence like for the complex structure, we get two connected components in the global theory linked by complex conjugation.*

The previous proposition shows that at any point, we are given a "generic" ideal whose Young diagram as described in 3.2 is just a column. Thus, we can think of a n-complex structure as a given *polynomial curve of order* $n-1$ *on each cotangent space* (or $(n-1)$-jet).

Before looking on the local and global theory, we can easily determine the global nature of the higher Beltrami coefficients. For this, let's see how they change under a holomorphic coordinate transform $z \to z(w)$. Since $p = \frac{\partial}{\partial z} \mapsto \frac{dw}{dz}\frac{\partial}{\partial w}$ and similarly $\bar{p} = \frac{\partial}{\partial \bar{z}} \mapsto \frac{d\bar{w}}{d\bar{z}}\frac{\partial}{\partial \bar{w}}$, we get

$$\begin{aligned}
&\left\langle p^n, -\bar{p} + \mu_2(z,\bar{z})p + ... + \mu_n(z,\bar{z})p^{n-1} \right\rangle \\
&\mapsto \left\langle (\frac{dw}{dz})^n (\frac{\partial}{\partial w})^n, -\frac{d\bar{w}}{d\bar{z}}\frac{\partial}{\partial \bar{w}} + \frac{dw}{dz}\mu_2(z,\bar{z})\frac{\partial}{\partial w} + ... + (\frac{dw}{dz})^{n-1}\mu_n(z,\bar{z})(\frac{\partial}{\partial w})^{n-1} \right\rangle \\
&= \left\langle (\frac{\partial}{\partial w})^n, -\frac{\partial}{\partial \bar{w}} + \frac{d\bar{z}/d\bar{w}}{dz/dw}\mu_2(z,\bar{z})\frac{\partial}{\partial w} + ... + \frac{d\bar{z}/d\bar{w}}{(dz/dw)^{n-1}}\mu_n(z,\bar{z})(\frac{\partial}{\partial w})^{n-1} \right\rangle
\end{aligned}$$

Thus, we see that for $m = 2, ..., n$ we get

$$\mu_m(w,\bar{w}) = \frac{d\bar{z}/d\bar{w}}{(dz/dw)^{m-1}}\mu_m(z,\bar{z}).$$

So μ_m is of type $(-m+1, 1)$, i.e. a section of $K^{-m+1} \otimes \bar{K}$ where $K = T^{*(1,0)}\Sigma$ is the canonical line bundle. For $m = 2$, this coincides with our observation in 2.3 on the global nature of the Beltrami coefficient.

Now, we go to the second step, the local theory. In the next subsection, we will explain why we have to enlarge our attention from Σ to the symplectic manifold $T^*\Sigma$.

4.2 Higher diffeomorphisms

In the previous section, we saw that the n-complex structure in one point can be seen as a polynomial curve of degree $n-1$ in the complexified cotangent space or equivalently as a complex-valued polynomial function on the (real) cotangent space. Thus, it seems clear that we cannot get $\mu_2 = ... = \mu_n = 0$ by a linear coordinate change. The best we can do is $\mu_2 = 0$ which corresponds to the fact that the almost complex structure can be trivialized at one point.

We want the higher complex structure to be as close to the complex structure as possible. In particular we wish to be able to trivialize it at one point. So what we need are polynomial transformations in the cotangent space. This cannot be achieved from a transformation on Σ alone, so we have to consider the whole manifold $T^*\Sigma$ which carries a natural symplectic structure. Why symplectic geometry?

Because any function H on a symplectic manifold generates a transformation, so we hope that this transformation will be polynomial when we choose H polynomial.

So we are entering the realm of **symplectic geometry**. Recall that a **symplectic structure** on a manifold is a closed non-degenerate differential 2-form ω. Darboux's theorem asserts that a symplectic manifold can be modeled over $(\mathbb{R}^{2n}, \omega_0)$ where $\omega_0 = dx_1 \wedge dx_2 + ... + dx_{2n-1} \wedge dx_{2n}$ is the standard symplectic form on $\mathbb{R}^{2n}$. The global theory of symplectic structures is not well-understood yet (especially there is no analog to theorem 2).

Any cotangent bundle admits a natural symplectic structure coming from the exterior derivative of the Liouville form. In our case, the symplectic form is simply given by $\omega = dp \wedge dz + d\bar{p} \wedge d\bar{z}$. Notice that although written in complexified coordinates, ω lives in the real cotangent space since $T^*\Sigma = \{adz + \bar{a}d\bar{z} \mid a \in \mathbb{C}\} \subset T^{*\mathbb{C}}\Sigma$.

A **symplectomorphism** of a symplectic manifold (M, ω) is a diffeomorphism preserving the symplectic form ω. The set of all symplectomorphisms, denoted by $\mathrm{Symp}(M)$ is an infinite-dimensional Lie group. Its Lie algebra is given by the set of **symplectic vector fields** where a vector field X is called symplectic if the 1-form $i_X\omega := \omega(X, .)$ is closed. If $i_X\omega = dH$ is exact, we speak of a **Hamiltonian vector field** with Hamiltonian H which is nothing else than a smooth function on M. Since any closed form is locally exact, any symplectic vector field is locally Hamiltonian.

Definition 6. *A **higher diffeomorphism** of a surface Σ is a symplectomorphism of $T^*\Sigma$ preserving the zero-section $\Sigma \subset T^*\Sigma$ (not necessarily pointwise). The set of higher diffeomorphisms is denoted by* $\mathrm{Symp}_0(T^*\Sigma)$.
*We say that a higher diffeomorphism is **of order** n if it is generated by a symplectic vector field such that around each point, its Hamiltonian H is a homogenous polynomial of degree n in p and $\bar{p}$.*

Some explanations seem necessary: For order $n = 1$, we get the usual diffeomorphisms of Σ, linearly extended to $T^*\Sigma$ since locally, the vector field dH is constant. In the usual situation, we have a coordinate z on Σ which gives a *linear coordinate* p on $T^*\Sigma$. A higher diffeomorphism distorts this linear coordinate (figure 7).

Further, a higher diffeomorphism generated by a symplectic vector field on $T^*\Sigma$ preserves the base iff the symplectic vector field restricted to the zero-section Σ lives in $T\Sigma \subset TT^*\Sigma$. Writing down this condition shows that *the Hamiltonian giving locally the symplectic vector field admits a Taylor development in p and $\bar{p}$ only.*

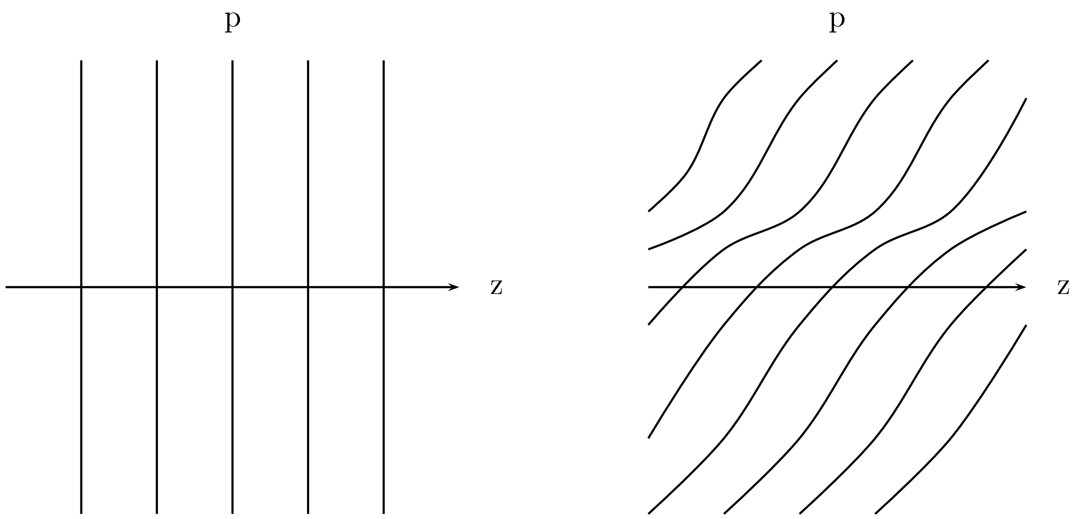

Fig. 7: Effect of higher diffeomorphisms

Let's see how a higher diffeomorphism acts on the n-complex structure. First, we need a general statement on variations of ideals:

Proposition 8. *The space of infinitesimal variations of an ideal I in a ring A is the set of all A-module homomorhpisms from I to A/I.*

Proof. Let $F : I \to A$ be an additive map such that $F(I)$ is an ideal. The condition for being an ideal gives that for all $a \in A$ and $x \in I$, there is a $z \in I$ with

$$aF(x) = F(z).$$

For an infinitesimal F, we can write $F = \mathrm{id} + \varepsilon f$. So we get

$$ax + \varepsilon a f(x) = z + \varepsilon f(z).$$

So $z = ax$ by taking $\varepsilon = 0$ and thus

$$af(x) = f(z) = f(ax)$$

which shows that f is an A-module homomorphism. Conversely, any such morphism gives a variation.
Since we do not change I when F stays in I, we have to consider morphisms modulo I. □

Remark. *In our case, we deal with ideals of codimension n of $\mathbb{C}[x, y]$ which is an algebra so ideals are vector spaces. Variations of vector spaces of constant dimension or codimension are described by the tangent space of a Grassmannian: Around a* *linear subspace L of a vector space V, the tangent space to the Grassmannian at L is given by all linear transformations from L to V/L.*

So to compute the variation of an ideal, all we need is to *compute the variation of its generators modulo I*. These generators are polynomial functions and symplectomorphisms act on functions. This gives the action of higher symplectomorphisms on the n-complex structure. We can explicite even further the infinitesimal variation by a Hamiltonian H.

Any Hamiltonian H generates a flow on $T^*\Sigma$ by integrating its associated vector field $X_H := \omega(dH, .)$. The variation of a function f along a flow line is given by

$$\frac{df}{dt} = df(X_H) = \{H, f\}$$

where $\{.,.\} := \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{p}} \wedge \frac{\partial}{\partial \bar{z}}$ is the **Poisson bracket** which is a 2-vector. This means that

$$\{H, f\} = \frac{\partial H}{\partial p}\frac{\partial f}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial f}{\partial p} + \frac{\partial H}{\partial \bar{p}}\frac{\partial f}{\partial \bar{z}} - \frac{\partial H}{\partial \bar{z}}\frac{\partial f}{\partial \bar{p}}.$$

So the infinitesimal variation of $I(z)$ under a Hamiltonian H is given by a function $F : I \to \mathbb{C}[x,y]/I$ defined by $p^n \mapsto \{H, p^n\} \mod I$ and

$$-\bar{p} + \mu_2(z,\bar{z})p + ... + \mu_n(z,\bar{z})p^{n-1} \mapsto \{H, -\bar{p} + \mu_2(z,\bar{z})p + ... + \mu_n(z,\bar{z})p^{n-1}\} \mod I$$

If we write out the Taylor expansion of H in p and $\bar{p}$, we see that only terms of degree at most $n-1$ will count, so we can assume H to be a sum of higher diffeomorphisms of order at most $n-1$.

In the next section, we will see how to compute this variation and that higher diffeomorphisms can trivialize locally the n-complex structure.

Remark. *It seems that higher diffeomorphisms have a strong link to* ***linear differential operators*** *on manifolds: usual diffeomorphisms isotopic to the identity are generated by vector fields which are nothing else than differential operators of order 1. It seems that higher diffeomorphisms isotopic to the identity are generated by linear differential operators since both symplectic vector fields and differential operators are given locally by a polynomial.*

4.3 Local theory

In this subsection, we are in an open neighborhood of 0 in $\mathbb{C}$. We will prove that a higher complex structure can locally be trivialized by a higher diffeomorphism, which means that $\mu_2(z,\bar{z}) = ... = \mu_n(z,\bar{z}) = 0$ in a neighborhood of 0. Before doing so, we have to compute the variation of the higher complex structure by a higher diffeomorphism.

4.3.1 Infinitesimal variation

Let's start with a small case before giving the general computation. For $n = 3$, we have an ideal of the form

$$I(z) = \left\langle p^3, -\bar{p} + \mu(z,\bar{z})p + \rho(z,\bar{z})p^2 \right\rangle$$

where $\mu = \mu_2$ is the usual Beltrami coefficient and $\rho = \mu_3$. Since we work locally, the infinitesimal variation is generated by a real Hamiltonian H of degree at most 2 in p and $\bar{p}$. We distinguish two cases:

<u>First case:</u> $H(z,\bar{z},p,\bar{p}) = v(z,\bar{z})p + \bar{v}(z,\bar{z})\bar{p}$ is of degree 1
As we have seen at the end of the previous subsection, the infinitesimal variation of $I(z)$ is determined by $\{H, -\bar{p} + \mu p + \rho p^2\} \mod I(z)$. We have

$$\begin{aligned}\{vp + \bar{v}\bar{p}, -\bar{p} + \mu p + \rho p^2\} = & -(\mu\partial\bar{v} - \bar{\partial}\bar{v})\bar{p} + (v\partial\mu - \mu\partial v + \bar{v}\bar{\partial}\mu + \bar{\partial}v)p \\ & +(v\partial\rho - 2\rho\partial v + \bar{v}\bar{\partial}\rho)p^2 - 2\rho\partial\bar{v}p\bar{p}\end{aligned}$$

The appearance of the term $p\bar{p}$ seems at first annoying but modulo I, we have $p\bar{p} = \mu p^2$. Now, integrating the Hamiltonian up to time ε in order to compute the variation of μ and ρ, we get working modulo I that

$$\begin{aligned}-\bar{p} + \mu p + \rho p^2 \mapsto & -(1 + \varepsilon(\mu\partial\bar{v} - \bar{\partial}\bar{v}))\bar{p} + (\mu + \varepsilon(v\partial\mu - \mu\partial v + \bar{v}\bar{\partial}\mu + \bar{\partial}v))p \\ & +(\rho + \varepsilon(v\partial\rho - 2\rho\partial v + \bar{v}\bar{\partial}\rho - 2\mu\rho\partial\bar{v}))p^2 \\ \propto & -\bar{p} + (\mu + \varepsilon(v\partial\mu - \mu\partial v + \bar{v}\bar{\partial}\mu + \bar{\partial}v - \mu^2\partial\bar{v} + \mu\bar{\partial}\bar{v})p \\ & +(\rho + \varepsilon(v\partial\rho - 2\rho\partial v + \bar{v}\bar{\partial}\rho - 3\mu\rho\partial\bar{v} + \rho\bar{\partial}\bar{v})p^2\end{aligned}$$

Thus, noticing a factorization we get

$$\delta\mu = (\bar{\partial} - \mu\partial + \partial\mu)(v + \mu\bar{v})$$

as we already computed in the remark at the end of subsection 2.4 (equation (4)) and

$$\delta\rho = v\partial\rho + \bar{v}\bar{\partial}\rho + \rho\bar{\partial}\bar{v} - 2\rho\partial v - 3\mu\rho\partial\bar{v}$$

We see that for a usual diffeomorphism (i.e. of order 1), we get the usual variation of the complex structure and around $\rho = 0$, we have $\delta\rho = 0$ so it remains zero.

Second case: $H = wp^2 + w'p\bar{p} + \bar{w}\bar{p}^2$ is of degree 2 with $w' \in \mathbb{R}$
We proceed exactly in the same manner as before. First we compute

$$\{wp^2 + w'p\bar{p} + \bar{w}\bar{p}^2, -\bar{p} + \mu p + \rho p^2\}$$

which equals

$$(2w\partial\mu - \mu\partial w + w'\bar{\partial}\mu + \bar{\partial}w)p^2 + (w'\partial\mu - \mu\partial w' + 2\bar{w}\bar{\partial}\mu + \bar{\partial}w')p\bar{p} + (\bar{\partial}\bar{w} - \mu\partial\bar{w})\bar{p}^2$$

Modulo I, we have $p\bar{p} = \mu p^2$ and $\bar{p}^2 = \mu^2 p^2$. We then get directly $\delta\rho$ since degree 1 terms are not affected by H:

$$\delta\rho = (2w\partial\mu - \mu\partial w + w'\bar{\partial}\mu + \bar{\partial}w) + \mu(w'\partial\mu - \mu\partial w' + 2\bar{w}\bar{\partial}\mu + \bar{\partial}w') + \mu^2(\bar{\partial}\bar{w} - \mu\partial\bar{w})$$

which beautifully factorizes to

$$\delta\rho = (\bar{\partial} - \mu\partial + 2\partial\mu)(w + \mu w' + \mu^2\bar{w}) \tag{11}$$

A Hamiltonian of degree at least 3 does not change μ and ρ. So we covered all cases.

We see in that example the importance to pass to higher diffeomorphisms: with a diffeomorphism, we cannot change ρ as we want but with a Hamiltonian of degree 2 we can. Around $\mu = 0$ for example, we get

$$\delta\rho = \bar{\partial}w$$

which is quite similar to the formula $\delta\mu = \bar{\partial}v$ for a Hamiltonian of degree 1.

We turn now to the general case. Equations (4) and (11) for the variations of $\mu = \mu_2$ and $\rho = \mu_3$ suggest the following general formula:

Proposition 9. *The variation of μ_{k+1} under a Hamiltonian of degree k given by $H = w_k p^k + w_{k-1}p^{k-1}\bar{p} + ... + w_0\bar{p}^k$ with $\bar{w}_l = w_{k-l}$ for all l is given by*

$$\delta\mu_{k+1} = (\bar{\partial} - \mu_2\partial + k\partial\mu_2)(w_k + \mu_2 w_{k-1} + \mu_2^2 w_{k-2} ... + \mu_2^k w_0) \tag{12}$$

In addition, around $(\mu_2, ..., \mu_n) = (0, ..., 0)$, we get $\delta\mu_l = 0$ for $l \neq k+1$.

Proof. There is no mystery: we repeat the same computation as for $n = 3$ in the general case. The hurried reader can skip these computations.
A simplification comes from the fact that we are only interested in the variation of μ_k for the moment and that only terms of order at least k are modified. Hence, we can concentrate on terms of degree exactly k. With this and by noting $w_{-1} = 0$, we

first compute

$$\{w_k p^k + w_{k-1}p^{k-1}\bar{p} + ... + w_0\bar{p}^k, -\bar{p} + \mu_2 p + ... + \mu_n p^{n-1}\} =$$
$$\sum_{m=0}^{k} p^{k-m}\bar{p}^m((k-m)w_{k-m}\partial\mu_2 - \mu_2\partial w_{k-m} + (m+1)w_{k-m-1}\bar{\partial}\mu_2 + \bar{\partial}w_{k-m})$$
$$+ \text{ higher terms}$$

Modulo I, we have $p^i\bar{p}^j = \mu^j p^{i+j} +$ higher terms for all i and j. Integrating the Hamiltonian up to time ε gives modulo higher terms that to the function $-\bar{p}+\mu_2 p+ ... + \mu_n p^{n-1}$ is added a term

$$\varepsilon\sum_{m=0}^{k}\mu_2^m((k-m)w_{k-m}\partial\mu_2 - \mu_2\partial w_{k-m} + (m+1)w_{k-m-1}\bar{\partial}\mu_2 + \bar{\partial}w_{k-m})p^k$$

Thus, noticing a factorization, we get

$$\begin{aligned}\delta\mu_{k+1} =& \sum_{m=0}^{k}\mu_2^m((k-m)w_{k-m}\partial\mu_2 - \mu_2\partial w_{k-m} + (m+1)w_{k-m-1}\bar{\partial}\mu_2 + \bar{\partial}w_{k-m}) \\ =&(\bar{\partial} - \mu_2\partial + k\partial\mu_2)(w_k + \mu_2 w_{k-1} + \mu_2^2 w_{k-2} + ... + \mu_2^k w_0)\end{aligned}$$

which gives equation (12).

Around $\mu_2 = ... = \mu_n = 0$, we get easily

$$\begin{aligned}-\bar{p} \mapsto& -\bar{p} + \varepsilon\sum_{m=0}^{k} p^{k-m}\bar{p}^m\bar{\partial}w_{k-m} \\ =& -\bar{p} + \varepsilon p^k\bar{\partial}w_k\end{aligned}$$

to order 1. Hence, $\delta\mu_l = 0$ for $l \neq k+1$ around $\mu_2 = ... = \mu_n = 0$. □

Note that around $\mu_2 = 0$, the higher complex structure is preserved by any holomorphic higher diffeomorphism. So the group which locally preserves the structure is infinite-dimensional.

4.3.2 Local triviality

We are now ready to give the local theory of the higher complex structure:

Theorem 8. *The n-complex structure can be locally trivialized, i.e. there is a higher diffeomorphism which sends the structure to $(\mu_2(z,\bar{z}), ..., \mu_n(z,\bar{z})) = (0, ..., 0)$ for all small z.*

Proof. The proof will be by induction. For $n = 2$, we already know the result which is Gauss' theorem on the existence of isothermal coordinates (theorem 1). So suppose that the statement is true for $n \geq 2$ and we will show it for $n+1$.

By induction hypothesis, there is a higher diffeomorphism which makes $\mu_2(z) = ... = \mu_n(z) = 0$ for all z near the origin. We will construct a higher diffeomorphism generated by a Hamiltonian of degree n giving $\mu_{n+1}(z) = 0$ for all z near 0. Since a Hamiltonian of degree n does not affect the μ_k with $k \leq n$ (see previous proposition), we are done.

Let's try a Hamiltonian of the form

$$H(z, \bar{z}, p, \bar{p}) = w_n(z, \bar{z}, p, \bar{p})p^n + \bar{w}_n(z, \bar{z}, p, \bar{p})\bar{p}^n$$

generating a flow ϕ_t. We denote by $\mu_{n+1}^t(z, \bar{z})$ the image of $\mu_{n+1}(z, \bar{z})$ by ϕ_t (note that ϕ_t fixes the zero-section pointwise). The variation formula (12) for $\mu_2 = 0$ then reads

$$\frac{d}{dt}\mu_{n+1}^t(z, \bar{z}) = \bar{\partial} w_n(z, \bar{z}, 0, 0)$$

Thus, the variation does not depend on time. We wish to have

$$\frac{d}{dt}(\mu_{n+1}^t(z, \bar{z})) = -\mu_{n+1}^{t=0}(z, \bar{z}).$$

So we have to solve

$$\bar{\partial} w_n(z, \bar{z}, 0, 0) = -\mu_{n+1}^0(z, \bar{z})$$

The inversion of the Cauchy-Riemann operator $\bar{\partial}$ is well-known. We denote its inverse by T. Explicitly, we have

$$Tf(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

for any square-integrable function f.
Therefore, on the zero-section we set $w_n(z, \bar{z}, 0, 0) = -T\mu_{n+1}^0(z, \bar{z})$ (since μ_{n+1} is smooth, it is locally square-integrable). To define it everywhere, we choose a bump function β, in our case a function on $\mathbb{C}^2$ which is 1 in a neighborhood of the origin and 0 outside a bigger neighborhood of the origin, and we put

$$w_n(z, \bar{z}, p, \bar{p}) = -\beta(p, \bar{p})T\mu_{n+1}^0(z, \bar{z}).$$

So the Hamiltonian is defined everywhere and gives a compactly supported vector field which therefore can be integrated for all times. We then get

$$\mu^t_{n+1}(z,\bar{z}) = (1-t)\mu^0_{n+1}(z,\bar{z})$$

Therefore, at time $t = 1$, μ_{n+1} vanishes everywhere. □

Remark. *One might wonder how it is possible to change $\mu_k(z,\bar{z})$ by a higher diffeomorphism which fixes the zero-section pointwise. In fact, the higher Beltrami coefficients do not only depend on z and $\bar{z}$ but also on the "non-linear torsion of p". That means the following: the chart z gives a linear coordinate p_{lin} on the cotangent space. The μ_k also depend on the derivatives of p (and $\bar{p}$) with respect to p_{lin} (and $\bar{p}_{\text{lin}}$).*

4.4 Geometric higher Teichmüller space

In this final subsection, we will discuss the global theory of the n-complex structure. We will define a generalization of the Teichmüller space, show that it is a contractible ball of dimension $(n^2-1)(g-1)$, where g denotes the genus of the surface, and describe its tangent and cotangent space.

Definition 7. *The space of all compatible n-complex structures modulo higher diffeomorphisms isotopic to the identity is called the* ***geometric higher Teichmüller space*** *and denoted by $\hat{\mathcal{T}}^n_\Sigma$.*

Since a higher diffeomorphism of order 1 is a usual diffeomorphism, we recover for $n = 2$ the usual Teichmüller space:

$$\hat{\mathcal{T}}^2_\Sigma = \mathcal{T}_\Sigma.$$

Our main result of the global theory is

Theorem 9. *For a surface Σ of genus $g \geq 2$ the geometric higher Teichmüller space $\hat{\mathcal{T}}^n_\Sigma$ is a contractible manifold of complex dimension $(n^2-1)(g-1)$. In addition, its cotangent space at any point $\mu = (\mu_2, ..., \mu_n)$ is given by*

$$T^*_\mu \hat{\mathcal{T}}^n_\Sigma = \bigoplus_{m=2}^{n} H^0(\Sigma, K^m)$$

Proof. Contractibility is quite easy as for the complex structure: given a n-complex structure $(\mu_2, ..., \mu_n)$ with $|\mu_2| < 1$, we can retract it via $(1-t)(\mu_2, ..., \mu_n)$ to $(0, ..., 0)$.

To see that it is a manifold, we will examine the infinitesimal variation around any point. This will also give a description of the tangent and cotangent space, as for the complex structure. By definition, we have

$$\hat{\mathcal{T}}^n_\Sigma = \{(\mu_2, ..., \mu_n) \mid \mu_m \in K^{-m+1} \otimes \bar{K}\ \forall\, m \text{ and } |\mu_2| < 1\} / \operatorname{Symp}_0(T^*\Sigma)$$

The infinitesimal variation around $\mu = (\mu_2, ..., \mu_n)$ is then given by

$$T_\mu \hat{\mathcal{T}}^n_\Sigma = \{(\delta\mu_2, ..., \delta\mu_n) \mid \delta\mu_m \in K^{-m+1} \otimes \bar{K}\ \forall\, m\} / \Gamma_{\operatorname{Symp}_0}(T^*\Sigma)$$

where $\Gamma_{\operatorname{Symp}_0}(T^*\Sigma)$ denotes the symplectic vector fields on $T^*\Sigma$ tangent to the zero-section.

In the previous subsection, we have seen that every n-complex structure is locally trivializable. So there is an atlas in which $(\mu_2, ..., \mu_n) = \mu = 0$. In addition we have computed the action of a symplectic vector field on the n-complex structure around $\mu = 0$. Locally, we can decompose its Hamiltonian into homogenous parts of degree 1 to $n-1$. All higher terms do not affect the n-complex structure. By proposition 6 (with $\mu_2 = 0$), we get

$$T_\mu \hat{\mathcal{T}}^n_\Sigma = \{(\delta\mu_2, ..., \delta\mu_n)\} / (\bar{\partial} w_1, ..., \bar{\partial} w_{n-1})$$

where w_m is a section of K^m. Thus, the tangent space splits into parts

$$T_\mu \hat{\mathcal{T}}^n_\Sigma = \{\delta\mu_2 \in \bar{K} \otimes K^{-1}\} / \bar{\partial} w_1 \oplus ... \oplus \{\delta\mu_n \in K^{-n+1} \otimes \bar{K}\} / \bar{\partial} w_{n-1}$$

To compute the cotangent space, we proceed in the same way as for the complex structure. We get

$$\begin{aligned}(\{\delta\mu_m\} / \bar{\partial} w_{m-1})^* =& \operatorname{Ann}(\bar{\partial} w_{m-1}) \\ =& \{t_m \in K^m \mid \textstyle\int t_m \bar{\partial} w_{m-1} = 0\ \forall\, w_{m-1} \in K^{m-1}\} \\ =& \{t_m \in K^m \mid \textstyle\int \bar{\partial} t_m w_{m-1} = 0\ \forall\, w_{m-1} \in K^{m-1}\} \\ =& \{t_m \in K^m \mid \bar{\partial} t_m = 0\} \\ =& H^0(\Sigma, K^m)\end{aligned}$$

Therefore

$$T^*_\mu \hat{\mathcal{T}}^n_\Sigma = \bigoplus_{m=2}^{n} H^0(\Sigma, K^m)$$

Now, we can compute the dimension of the geometric higher Teichmüller space using the Riemann-Roch formula and Serre duality:

$$\dim H^0(\Sigma, K^m) = \deg K^m - g + 1 + \dim H^0(\Sigma, K^{-m+1}).$$

First, we have $\deg K^m = m \deg K = m(2g-2)$. For the term $\dim H^0(\Sigma, K^{-m+1})$, which describes the global holomorphic sections of K^{-m+1}, we have for $g \geq 2$ that $\deg K^{-m+1} = (1-m)(2g-2) < 0$. Since the degree is the number of zeros minus the number of poles for any meromorphic section, there cannot be any non-zero holomorphic section. Hence, $\dim H^0(\Sigma, K^{-m+1}) = 0$ and

$$\dim H^0(\Sigma, K^m) = (2m-1)(g-1).$$

Therefore

$$\dim \hat{\mathcal{T}}^n_\Sigma = \dim T^*_\mu \hat{\mathcal{T}}^n_\Sigma = \sum_{m=2}^{n} \dim H^0(K^m) = \sum_{m=2}^{n} (2m-1)(g-1) = (n^2-1)(g-1)$$

□

Remark. *We see that as in the case for the complex structure, the tangent space to the geometric higher Teichmüller space is the direct sum of the cokernels of the maps*

$$\bar{\partial} : \Omega^0(\Sigma, K^m) \to \Omega^{0,1}(\Sigma, K^m).$$

The proof of the previous theorem is not very difficult once the computations above are done. Even these are straightforward. The same theorem holds also for the (algebraic) higher Teichmüller space $\mathcal{T}^n_\Sigma$ discussed in section 2.5 which is considered as a difficult theorem (see [16]). So if we could prove the equivalence between the geometric and the algebraic higher Teichmüller space, these properties would get an easier proof. Furthermore, if both concepts are isomorphic, there would be a natural action of the mapping class group of the surface on $\mathcal{T}^n_\Sigma$. Also there would be some hope of finding a hyperkähler structure in the neighborhood of the zero-section of $T^*\mathcal{T}^n_\Sigma$ by the hyperkähler quotient construction described in [17]. In particular, this would give a Kähler structure on $\mathcal{T}^n_\Sigma$.

References

[1] Abikoff, William: *The uniformization theorem*, American Math. Monthly, Vol. 88, No. 8 (1981), pp. 574-592

[2] Bach, Samuel and Brunerie, Guillaume: *De l'espace de Teichmüller des surfaces*, pedagogical archive of ENS Paris, 2009-2010

[3] Baez, John and Muniain, Javier P.: *Gauge fields, Knots and Gravity*, Series on Knots and Everything Vol.4, World Scientific, 1994

[4] Baranovsky, Vladimir: *The Variety of Pairs of Commuting Nilpotent Matrices is Irreducible*, Transformation Groups, Vol.6, No.1 (2001), pp. 3-8

[5] Bejleri, Dori: *Hilbert schemes: Geometry, Combinatorics and Representation Theory*, notes for the Brown Graduate Student Seminar, http://www.math.brown.edu/∼dbejleri/Hilbert%20Schemes%20-%20Grad%20Student%20Seminar.pdf

[6] Chern, Shiing-Shen: *An elementary proof of the existence of isothermal parameters on a surface*, Proc. AMS Vol. 6, No. 5 (1955), pp. 771-782

[7] Donaldson, Simon K.: *Riemann surfaces*, Oxford Graduate Texts for Mathematics, 2011

[8] Dumas, David: *Complex projective structures*, http://homepages.math.uic.edu/∼ ddumas/work/survey/survey.pdf

[9] Ene, Viviana and Herzog, Jürgen: *Gröbner Bases in Commutative Algebra*, Grad. Stud. in Math. Vol.130, AMS, 2012

[10] Fock, Vladimir and Goncharow, Alexander: *Dual Teichmüller and lamination spaces*, arXiv, http://arxiv.org/pdf/math/0510312.pdf

[11] Fogarty, John: *Algebraic families on an algebraic surface*, Amer. J. Math. Vol. 90 No. 2 (1968), pp. 511-521

[12] Goldman, William M.: *Topological Components of Spaces of Representations*, Invent. Math. 93 (1988), pp. 557-607

[13] Griffiths, Phillip and Harris, Joseph: *Principles of Algebraic Geometry*, Wiley-Interscience, 1994

[14] Guichard, Olivier and Wienhard, Anna: *Convex Foliated Projective Structures and the Hitchin Component for* $PSL(4,\mathbb{R})$, Duke Math. Journal 144, No. 3 (2008), pp. 381-445

[15] Haiman, Mark: t, q*-Catal numbers and the Hilbert scheme*, Discrete Mathematics 193 (1998), pp. 201-224

[16] Hitchin, Nigel: *Lie Groups and Teichmüller Space*, Topology Vol.31, No.3 (1992), pp. 449-473

[17] Hitchin, Nigel: *Hyperkähler Manifolds*, Séminaire N. Bourbaki, 1991-1992, exp. n^o 748, pp. 139-166

[18] Iarrobino, Anthony: *Punctual Hilbert schemes*, Bull. AMS Vol. 78, No. 5 (1972), pp. 819-823

[19] Johnson, Warren P.: *The Curious History of Faà di Bruno's formula*, The American Mathematical Monthly, vol. 109, 2002, pp. 217-234

[20] Kodaira, Kunihiko: *Complex Manifolds and Deformation of Complex Structures*, Classics in Mathematics, Springer-Verlag, 2005

[21] Lee, John M.: *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics Vol. 218, Springer-Verlag, 2012

[22] McDuff, Dusa and Salamon, Dietmar: *Introduction to Symplectic Topology*, Clarendon Press, 1998

[23] Morse, Philip M. and Feshbach, Herman: *Methods of theoretical physics, Part I*, New York: McGraw-Hill, 1953, pp. 411-413

[24] Nakajima, Hiraku: *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series 18, AMS, 1999

[25] Poincaré, Henri: *Sur l'uniformisation des fonctions analytiques*, Acta Mathematica 31, pp. 1-63